ON THE MOTION OF VORTEX SHEETS WITH SURFACE TENSION IN THE 3D EULER EQUATIONS WITH VORTICITY

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1. Introduction

The motion of vortex sheets with surface tension has been analyzed in the setting of irrotational flows by Ambrose [1] and Ambrose & Masmoudi [2] in 2D, and by Ambrose & Masmoudi [3] in 3D. With irrotationality, the nonlinear Euler equations reduce to the Laplace equation for the pressure function in the bulk, and the motion of the vortex sheet is decoupled from that of the fluid, thus allowing boundary integral methods to be employed. In a general flow with vorticity, the full two-phase Euler equations must be analyzed; in this situation, the vortex sheet is a surface of discontinuity representing the material interface between two incompressible inviscid fluids with densities ρ^+ and ρ^-, respectively. The tangential velocity of the fluid suffers a jump discontinuity along the material interface, leading to the well-known Kelvin-Helmholtz or Rayleigh-Taylor instabilities when surface tension is neglected. The velocity of the vortex sheet is the normal component of the fluid velocity, whose continuity across the material interface $\Gamma(t)$ is enforced. In addition to incompressibility, the continuity of the normal component of velocity across $\Gamma(t)$ is a fundamental difference between multi-D shock wave evolution, wherein the velocity of the surface of discontinuity is determined by the generalized Rankine-Hugoniot condition. Nevertheless, the problems are mathematically very similar, and we refer the reader to the book of Majda [6] for the analysis of multi-D shocks.

In the incompressible, *rotational* flow-setting, very little analysis has been made of the two-phase Euler equations. With surface tension present, Shatah and Zeng [7] have obtained formal a priori estimates for smooth enough solutions, but the question of existence of smooth solutions remains open. In this paper, following the methodology of Coutand & Shkoller [4], we prove well-posedness for short-time for this problem.

Let Ω^+ and Ω^- denote two open bounded subsets of $\mathbb{R}^3$ such that $\Omega = \Omega^+ \cup \Omega^-$ denotes the total volume occupied by the two fluids, and $\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$ denotes the material interface. We assume that it is the region $\overline{\Omega^-}$ that intersects $\partial\Omega$.

Let η denote the Lagrangian flow map, satisfying

$$\begin{aligned} &\eta_t(x,t) = u(\eta(x,t),t) \qquad \forall\ x \in \Omega, t > 0\,, \\ &\eta(x,0) = x\,. \end{aligned}$$

Let $\Omega^+(t)$, $\Omega^-(t)$ and $\Gamma(t)$ denote $\eta(t)(\Omega^+)$, $\eta(t)(\Omega^-)$ and $\eta(t)(\Gamma)$, respectively, and let $u^\pm$ and $p^\pm$ denote the velocity field and pressure function, respectively, in $\Omega^\pm(t)$.

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The incompressible Euler equations for the motion of two fluids can be written as

$$\rho^\pm(u_t^\pm + \nabla_{u^\pm} u^\pm) + \nabla p^\pm = 0 \qquad \text{in} \quad \Omega^\pm(t)\,, \tag{1.1a}$$

$$\operatorname{div} u^\pm = 0 \qquad \text{in} \quad \Omega^\pm(t)\,, \tag{1.1b}$$

$$[p]_\pm = \sigma H \qquad \text{on} \quad \Gamma(t)\,, \tag{1.1c}$$

$$[u \cdot n]_\pm = 0 \qquad \text{on} \quad \Gamma(t)\,, \tag{1.1d}$$

$$u^- \cdot n = 0 \qquad \text{on} \quad \partial\Omega\,, \tag{1.1e}$$

$$u(0) = u_0 \qquad \text{on} \quad \{t=0\} \times \Omega\,, \tag{1.1f}$$

where the material interface $\Gamma(t)$ moves with speed $u(t)^+ \cdot n(t)$, ρ^+ and ρ^- are the densities of the two fluids occupying $\Omega^+(t)$ and $\Omega^-(t)$, respectively, $H(t)$ is twice the mean curvature of $\Gamma(t)$, $\sigma > 0$ is the surface tension parameter, and $n(t)$ denotes the outward-pointing unit normal on $\partial\Omega^+(t)$.

THEOREM 1.1 (Main result). *Suppose that $\sigma > 0$, Γ is of class H^4, $\partial\Omega$ is of class H^3, and $u_0^\pm \in H^3(\Omega^\pm)$. Then, there exists $T > 0$, and a solution $(u^\pm(t), p^\pm(t), \Omega^\pm(t))$ of (1.1) with $u^\pm \in L^\infty(0,T; H^3(\Omega^\pm(t))$, $p^\pm \in L^\infty(0,T; H^{2.5}(\Omega^\pm(t))$, and $\Gamma(t) \in H^4$. The solution is unique if $u_0^\pm \in H^{4.5}(\Omega^\pm)$ and $\Gamma \in H^{5.5}$.*

The paper is organized as follows. In Section 2, we establish the notation to be used throughout the paper. In Section 3 we establish low-regularity trace theorems of the normal and tangential components of L^2 vector fields with divergence and curl structure. In Section 4, we introduce a regularized version of the Euler equations (1.1); the transport velocity and the domain are regularized using the tool of horizontal convolution by layers that we introduced in [4]. Additionally, a nonlinear parabolic regularization of the surface tension operator is made in the Laplace-Young boundary condition (4.1d). Section 5 is devoted to the existence of solutions to (4.1). In Section 6, we obtain estimates for the velocity, pressure, and their time derivatives at time $t = 0$. Section 7 provides the pressure estimates that we need for a priori estimates. In Section 8, we establish the κ-independent estimates for the solutions of the κ-problem (4.1); this allows us to pass to the limit as the regularization parameter $\kappa \to 0$ and prove existence of solutions to (1.1). In Section 9, we provide the optimal regularity requirements on the data. Finally, in Section 10 we prove uniqueness of solutions.

2. NOTATION

Let $\mathfrak{n} := \dim(\Omega) = 2$ or 3. We will use the notation $H^s(\Omega^+)$ ($H^s(\Omega^-)$) to denote either $H^s(\Omega^+;\mathbb{R})$ ($H^s(\Omega^-;\mathbb{R})$) for a scalar function or $H^s(\Omega^+;\mathbb{R}^\mathfrak{n})$ ($H^s(\Omega^-;\mathbb{R}^\mathfrak{n})$) for a vector valued function, and we denote the $H^s(\Omega^\pm)$-norm by

$$\|w\|_{s,+} = \|w\|_{H^s(\Omega^+)} \quad \text{and} \quad \|w\|_{s,-} = \|w\|_{H^s(\Omega^-)}\,.$$

The $H^s(\Gamma)$- and $H^s(\partial\Omega)$-norms are denoted by

$$|w|_s = \|w\|_{H^s(\Gamma)} \quad \text{and} \quad |w|_{s,\partial\Omega} = \|w\|_{H^s(\partial\Omega)}\,.$$

For simplicity, we also use $\|w\|^2_{s,\pm}$ and $|w|^2_{s,\pm}$ to denote $\|w^+\|^2_{s,+} + \|w^-\|^2_{s,-}$ and $|w^+|^2_s + |w^-|^2_s$, respectively. That is,

$$\|w\|^2_{s,\pm} = \|w^+\|^2_{s,+} + \|w^-\|^2_{s,-}\,,$$
$$|w|^2_{s,\pm} = |w^+|^2_s + |w^-|^2_s\,.$$

For $s \geq 1.5$, $\mathcal{V}^s_+(T)$ denotes the space

$$\Big\{ w \in L^2(0,T; L^2(\Omega^+)) \;\Big|\; w \in L^2(0,T; H^s(\Omega^+) \cap L^\infty(0,T; H^{s-1.5}(\Omega^+)) \Big\}$$

with associated norm

$$\|w\|_{\mathcal{V}^s_+(T)} = \sup_{t\in[0,T]} \|w(t)\|_{H^{s-1.5}(\Omega^+)} + \int_0^T \|w^+(s)\|^2_{H^s(\Omega^+)} ds\,,$$

where w can be either vector-valued or scalar-valued. The space $\mathcal{V}^s_-(T)$ is defined slightly differently, namely

$$\mathcal{V}^s_-(T) \equiv \Big\{ w \in L^2(0,T; L^2(\Omega^-)) \;\Big|\; w \in L^2(0,T; H^s(\Omega^-) \cap L^\infty(0,T; H^{s-1}(\Omega^-)) \Big\}$$

with norm

$$\|w\|_{\mathcal{V}^s_+(T)} = \sup_{t\in[0,T]} \|w(t)\|_{H^{s-1}(\Omega^-)} + \int_0^T \|w(s)\|^2_{H^s(\Omega^-)} ds\,.$$

As in [4], the energy function is defined as

$$E_\kappa(t) = \|\eta^+\|^2_{\mathfrak{n}+2.5,+} + \sum_{j=0}^{\mathfrak{n}} \|\partial_t^j v\|^2_{3.5-j,\pm} + \|\partial_t^{\mathfrak{n}+1} v\|^2_{0,\pm} + \|\sqrt{\kappa}\eta^+\|^2_{\mathfrak{n}+3.5,+} \tag{2.1}$$
$$+ \sum_{j=0}^{\mathfrak{n}+1} \int_0^T \|\sqrt{\kappa}\partial_t^j v^+\|^2_{4.5-j,+} dt\,.$$

We use the notation $f^\pm = g^\pm + h^+ + k^-$ to mean that

$$f^+ = g^+ + h^+\,, \text{ and } f^- = g^- + k^-\,.$$

3. Trace theorems

The normal trace theorem which states that the existence of the normal trace of a velocity field $w \in L^2(\Omega)$ relies on the regularity of $\operatorname{div} u$ (see, for example, [8]). If $\operatorname{div} w \in H^1(\Omega)'$, then $w \cdot N$, the normal trace, exists in $H^{-0.5}(\partial\Omega)$ so that

$$\|w \cdot N\|_{H^{-0.5}(\partial\Omega)} \leq C\Big[\|w\|^2_{L^2(\Omega)} + \|\operatorname{div} w\|^2_{H^1(\Omega)'}\Big] \tag{3.1}$$

for some constant C independent of w. In addition to the normal trace theorem, we have the following.

Theorem 3.1. *Let $w \in L^2(\Omega)$ so that $\operatorname{curl} w \in H^1(\Omega)'$, and let τ_1, τ_2 be a basis of the vector field on $\partial\Omega$, i.e., any vector field u can be uniquely written as $u^\alpha \tau_\alpha$. Then*

$$\|w \cdot \tau_\alpha\|_{H^{-0.5}(\partial\Omega)} \leq C\Big[\|w\|^2_{L^2(\Omega)} + \|\operatorname{curl} w\|^2_{H^1(\Omega)'}\Big]\,, \qquad \alpha = 1, 2 \tag{3.2}$$

for some constant C independent of w.

Proof. Given $\psi \in H^{0.5}(\partial\Omega)$, let $\phi_\alpha \in H^1(\Omega)$ be defined by

$$\begin{aligned} \Delta\phi_\alpha &= 0 && \text{in } \Omega\,, \\ \phi_\alpha &= (N \times \tau_\alpha)\psi && \text{on } \partial\Omega\,. \end{aligned}$$

Then

$$\int_{\partial\Omega} (w \cdot \tau_\alpha)\psi dS = \int_\Omega \operatorname{curl} w \cdot \phi_\alpha dx - \int_\Omega \operatorname{curl} \phi_\alpha \cdot w dx$$

and hence

$$\Big|\int_{\partial\Omega}(w\cdot\tau_\alpha)\psi dS\Big| \le C\Big[\|w\|^2_{L^2(\Omega)}+\|\operatorname{curl} w\|_{H^1(\Omega)'}\Big]\|\phi_\alpha\|_{H^1(\Omega)}$$
$$\le C\Big[\|w\|^2_{L^2(\Omega)}+\|\operatorname{curl} w\|_{H^1(\Omega)'}\Big]\|\psi\|_{H^{0.5}(\partial\Omega)}$$

which implies the desired inequality. □

Combining (3.1) and (3.2), we have the following:

$$\|w\|_{H^{-0.5}(\partial\Omega)} \le C\Big[\|w\|_{L^2(\Omega)}+\|\operatorname{div} w\|_{H^1(\Omega)'}+\|\operatorname{curl} w\|_{H^1(\Omega)'}\Big] \tag{3.3}$$

for some constant C independent of w.

4. The regularized κ-problem

Let Ω' be an open subset of Ω so that $\Omega^+ \subset\subset \Omega' \subset\subset \Omega$. In the following discussion, we will use $M^+ : H^{5.5}(\Omega^+) \to H^{5.5}(\Omega)$ to denote a fixed bounded extension operator (from the plus region to the whole region) so that $M^+v = 0$ in Ω'^c for all $v \in H^{5.5}(\Omega^+)$.

Let v^+ be the Lagrangian velocity in the plus region Ω^+, and $v_e = M^+v^+$ with v_κ defined as the horizontal convolution by layers of v_e. Let $\eta_\kappa = \mathrm{Id} + \int_0^t v_\kappa(s)ds$ be the Lagrangian coordinate (or flow map) of v_κ, and the Jacobian $\mathcal{J}_\kappa$, the cofactor matrices a_κ and the normal n_κ are defined accordingly.

The smoothed κ-problems is then defined as

$$\eta_e = \mathrm{Id} + \int_0^t v_e(s)ds \qquad \text{in } [0,T]\times\Omega^\pm\,, \tag{4.1a}$$
$$\rho^\pm \mathcal{J}_\kappa v_t^{\pm i} + (a_\kappa)^\ell_j(v^{-j}-v_e^{-j})v^{\pm i}_{,\ell} + (a_\kappa)^j_i q^\pm_{,j} = 0 \qquad \text{in } [0,T]\times\Omega^\pm\,, \tag{4.1b}$$
$$(a_\kappa)^j_i v^{\pm i}_{,j} = 0 \qquad \text{in } [0,T]\times\Omega^\pm\,, \tag{4.1c}$$
$$q^+ - q^- = -\sigma\Delta_g(\eta_e)\cdot n_\kappa - \kappa\Delta_0(v^+\cdot n_\kappa) \qquad \text{on } [0,T]\times\Gamma\,, \tag{4.1d}$$
$$v^+\cdot n_\kappa = v^-\cdot n_\kappa \qquad \text{on } [0,T]\times\Gamma\,, \tag{4.1e}$$
$$v^-\cdot n_\kappa = 0 \qquad \text{on } [0,T]\times\partial\Omega\,, \tag{4.1f}$$
$$v^\pm(0) = u_0^\pm \qquad \text{in } \{t=0\}\times\Omega^\pm\,, \tag{4.1g}$$

where $g_\kappa = \eta_{\kappa,1}\cdot\eta_{\kappa,2}$ is the induced metric on Γ. Note that since M^+ extends v^+ continuously to the whole domain Ω, $\eta^+_\kappa = \eta^-_\kappa$ and $\eta^+_e = \eta^-_e$ on Γ.

Remark 1. *Since $M^+v = 0$ in Ω'^c for all $v \in H^{5.5}(\Omega)$, $\bar a_\kappa = Id$ and $n_\kappa = N$ on $\partial\Omega$. Therefore, the boundary condition (4.1f) can also be written as $v^-\cdot N = 0$ where N denote the outward pointing unit normal of Ω^- on $\partial\Omega$.*

5. Existence of solutions for the regularized κ-problem

5.1. The iteration between the solution in Ω^+ and Ω^-. Let $(\bar v^+, \bar v^- \bar q) \in \mathcal{V}^{5.5}_+(T)\times\mathcal{V}^{4.5}_-(T)\times\mathcal{V}^{3.5}_-(T)/\mathbb{R}$ be given, and let $\bar v^+_\kappa$ be the horizontal convolution by layers of $\bar v^+$. Define $\bar v_e = M^+\bar v^+_\kappa$, the extension of $\bar v^+_\kappa$, with the associated Lagrangian map $\bar\eta_\kappa = \mathrm{Id} + \int_0^t \bar v_e(s)ds$ and cofactor matrix $\bar a_\kappa = \bar{\mathcal{J}}_\kappa(\nabla\bar\eta_\kappa)^{-1}$ where $\bar{\mathcal{J}}_\kappa = \det(\nabla\bar\eta_\kappa)$ is the Jacobian. The normal vector $\bar n_\kappa$ is then defined by

$$\bar n^i_\kappa = \bar g^{-\frac12}\varepsilon_{ijk}\bar\eta^j_{\kappa,1}\bar\eta^k_{\kappa,2} = \bar g^{-\frac12}(\bar a_\kappa)^j_i N_j\,.$$

The process of finding solutions to (4.1) consists of finding solutions to the following two problems. First, in the plus region Ω^+, we solve

$$\begin{aligned}
\rho^+\bar{\mathcal{J}}_\kappa w_t^i + (\bar{a}_\kappa)_i^j r_{,j} &= 0 && \text{in} \quad [0,T]\times\Omega^+\,, && (5.1a)\\
(\bar{a}_\kappa)_i^j w^i_{,j} &= 0 && \text{in} \quad [0,T]\times\Omega^+\,, && (5.1b)\\
r = \bar{q} - \sigma L_{\bar{g}}(\bar{\eta})\cdot\bar{n}_\kappa - \kappa\Delta_{\bar{0}}(w\cdot\bar{n}_\kappa) & && \text{on} \quad [0,T]\times\Gamma\,, && (5.1c)\\
w(0) &= u_0^+ && \text{on} \quad \{t=0\}\times\Omega\,, && (5.1d)
\end{aligned}$$

where $w = u^+\circ\bar{\eta}_\kappa$, $r = p^+\circ\bar{\eta}_\kappa$, $\bar{\eta} = \mathrm{Id} + \int_0^t \bar{v}(s)ds$ and $\Delta_{\bar{0}} = \bar{g}^{-\frac{1}{2}}\partial_\alpha[\sqrt{g_0}g_0^{\alpha\beta}\partial_\beta]$. Once the solution (w,r) to (5.1) is obtained, then in the minus region Ω^-, we solve

$$\begin{aligned}
\rho^-\Big[\bar{\mathcal{J}}_\kappa v_t^i + (\bar{a}_\kappa)_j^\ell(\bar{v}^- - \bar{v}_e^-)v^i_{,\ell}\Big] + (\bar{a}_\kappa)_i^j q_{,j} &= 0 && \text{in} \quad [0,T]\times\Omega^-\,, && (5.2a)\\
(\bar{a}_\kappa)_i^j v^i_{,j} &= 0 && \text{in} \quad [0,T]\times\Omega^-\,, && (5.2b)\\
v\cdot\bar{n}_\kappa &= w\cdot\bar{n}_\kappa && \text{on} \quad [0,T]\times\Gamma\,, && (5.2c)\\
v\cdot\bar{n}_\kappa &= 0 && \text{on} \quad [0,T]\times\partial\Omega\,, && (5.2d)\\
v(0) &= u_0^- && \text{on} \quad \{t=0\}\times\Omega\,, && (5.2e)
\end{aligned}$$

where $v = u^-\circ\bar{\eta}_\kappa$, $q = p^-\circ\bar{\eta}_\kappa$.

This process introduces the map $\Phi : (\bar{v}^+,\bar{v}^-,\bar{q}) \mapsto (w,v,q)$, and the fixed-points of Φ provides solutions to problem (4.1).

5.2. **Estimates for the solution in Ω^+.** The only difference between (5.1) and the one phase problem studied in [4] is the presence of the term $\bar{q}$ in the boundary condition (5.1c). We note that if $\bar{q}$ is smooth, then by exactly the same argument as in [4], the solution to (5.1) will be also be smooth, depending on the regularity of the initial velocity u_0^+. Therefore, for $\bar{q}$ given in $L^2(0,T;H^{3.5}(\Omega^-)/\mathbb{R})$, we replace (5.1c) by

$$r = \bar{q}_\epsilon - \sigma L_{\bar{g}}(\bar{\eta})\cdot\bar{n}_\kappa - \kappa\Delta_{\bar{0}}(w\cdot\bar{n}_\kappa) \quad \text{on} \quad \Gamma\,, \tag{5.3}$$

where $\bar{q}_\epsilon$ denotes the horizontal convolution by layers of $\bar{q}$. The solution w^ϵ and r^ϵ to (5.1a), (5.1b), (5.3) and (5.1d) are smooth functions satisfying

$$\|w^\epsilon\|^2_{0,+} + \int_0^t \Big[\|r^\epsilon\|^2_{3.5,+} + \kappa|w^\epsilon\cdot\bar{n}_\kappa|^2_{5,\pm}\Big]ds \le N(u_0) + C(\kappa,\bar{v}^+,\bar{q})\sqrt{t}\,, \tag{5.4}$$

where $C(\kappa,\bar{v}^+,\bar{q})$ denotes a constant depending on κ, $\|\bar{v}^+\|_{\mathcal{V}_+^{5.5}(T)}$, $\|\bar{q}\|_{\mathcal{V}_-^{3.5}(T)}$. Note that although this constant depends on ρ^+ as well, we omit this dependence in the estimate since it is a constant.

The divergence and curl estimates as in [4] can also be carried on so that

$$\|\operatorname{curl} w^\epsilon\|^2_{4.5,+} + \|\operatorname{div} w^\epsilon\|^2_{4.5,+} \le N(u_0) + C(\kappa,\bar{v}^+)\int_0^t \|w^\epsilon\|^2_{5.5,+}ds \tag{5.5}$$

for some constant $C(\kappa,\bar{v}^+)$ independent of the smooth parameter ϵ. Estimates (5.5) and (5.4) imply that

$$\int_0^t \|w^\epsilon(s)\|^2_{5.5,+}ds \le \frac{t}{\kappa}N(u_0) + C(\kappa,\bar{v}^+)\int_0^t\int_0^s \|w^\epsilon(s')\|^2_{5.5,+}ds'ds$$

and the Gronwall inequality implies that

$$\int_0^t \|w^\epsilon(s)\|^2_{5.5,+}ds \le C(\kappa,u_0^+,\bar{v}^+,\bar{q})\sqrt{t}\,. \tag{5.6}$$

By studying the elliptic problem for r^ϵ with the Dirichlet boundary condition (5.3), we find that

$$\int_0^t \|r^\epsilon(s)\|^2_{3.5,+} ds \le C(\kappa, u_0^+, \bar v^+, \bar q)\sqrt{t}\,. \tag{5.7}$$

(5.7) implies that $w_t^\epsilon \in L^2(0,T;H^{2.5}(\Omega^+))$ and by interpolations,

$$\sup_{t\in[0,T]} \|w^\epsilon(t)\|^2_{4,+} \le \|u_0^+\|^2_{4,+} + C(\kappa, u_0^+, \bar v^+, \bar q)\sqrt{T}\,. \tag{5.8}$$

(5.8) further implies that

$$\|r^\epsilon\|^2_{2,+} \le N(u_0) + C(\kappa,\delta, u_0^+, \bar v^+, \bar q)\sqrt{t} + \delta\|\bar q\|^2_{2,-}\,. \tag{5.9}$$

It also follows from (5.1a) that $\|w_t^\epsilon\|^2_{\mathcal{V}_+^{2.5}(T)}$ shares the same bound as $\|r^\epsilon\|^2_{\mathcal{V}_-^{3.5}(T)}$, i.e.,

$$\|w_t^\epsilon\|^2_{\mathcal{V}_+^{2.5}(T)} \le N(u_0) + C(\kappa,\delta, u_0^+, \bar v^+, \bar q)\sqrt{t} + \delta\|\bar q\|^2_{2,-}\,. \tag{5.10}$$

These ϵ independent estimates enable us to pass $\epsilon \to 0$ and obtained solution (w, r) to problem (5.1) with estimate

$$\|w\|^2_{\mathcal{V}_+^{5.5}(T)} + \|w_t\|^2_{\mathcal{V}_+^{2.5}(T)} + \|r\|^2_{\mathcal{V}_+^{3.5}(T)} \le N(u_0) + C_{\kappa,\delta}\sqrt{T} + \delta\|\bar q\|^2_{2,-}\,, \tag{5.11}$$

where $C_{\kappa,\delta}$ is the short hand notation for $C(\kappa,\delta, u_0^+, \bar v^+, \bar q)$.

5.3. **Estimates for the solution in Ω^-.** We will set up a iterative scheme in order to show the existence of a solution to problem (5.2). Let $\bar A^i_j = \bar{\mathcal{J}}_\kappa^{-1}(\bar a_\kappa)^i_j$. For a given $\bar w \in \mathcal{V}_-^{4.5}(T)$ with $\bar w_t \in \mathcal{V}_-^{2.5}(T)$, we solve first

$$\bar A^k_i[\bar A^j_i q_{,j}]_{,k} = -\,\rho^-\bar A^k_r \bar v^r_{\kappa,s}\bar A^s_i \bar w^i_{,k} - \rho^-\bar A^k_i[\bar A^\ell_j(\bar v^{-j} - \bar v_e^{-j})\bar w^i_{,\ell}]_{,k} \quad \text{in } \Omega^-\,, \tag{5.12a}$$

$$\bar A^j_i q_{,j}\bar n^i_\kappa = -\rho^-[w_t\cdot \bar n_\kappa + (w-\bar w)\cdot \bar n_{\kappa t} + \bar A^\ell_j(\bar v^{-j} - \bar v_e^{-j})\bar w^i_{,\ell}\bar n^i_\kappa] \quad \text{on } \Gamma\,, \tag{5.12b}$$

$$\bar A^j_i q_{,j}\bar n^i_\kappa = \rho^-[\bar w\cdot \bar n_{\kappa t} - \bar A^\ell_j(\bar v_\kappa^{-j} - \bar v_e^{-j})\bar w^i_{,\ell}\bar n^i_\kappa] \quad \text{on } \partial\Omega\,. \tag{5.12c}$$

Once a solution to (5.12) is obtained, use this solution q in (5.2a) and solve the transport equation

$$\begin{aligned} \rho^-\Big[\bar{\mathcal{J}}_\kappa v^i_t + (\bar a_\kappa)^\ell_j(\bar v^{-j} - \bar v_e^{-j})v^i_{,\ell}\Big] + (\bar a_\kappa)^j_i q_{,j} &= 0 && \text{in } \Omega^-\,,\\ v(0) &= u_0^- && \text{in } \Omega^-\,. \end{aligned}$$

Suppose we can prove that v is actually in the space we start with, then a fixed-point of the map $\Psi : \bar w \mapsto v$ provides a solution to problem (5.2).

We note that in this iterative scheme $\bar A$ is always fixed with estimates

$$\|\mathrm{Id} - \bar A(t)\|_{4.5,-} \le C(\bar v^+)\sqrt{t} \tag{5.13}$$

for some constant C depending on $\|\bar v^+\|^2_{\mathcal{V}_+^{5.5}(T)}$ but independent of κ. Therefore, by assuming that T is small enough (so that $C(\bar v^+)T$ is small), it follows from elliptic theory (see [5]) that

$$\begin{aligned} \|q\|^2_{3.5,-} &\le C\Big[\|\bar w\|^2_{3.5,-} + \|w_t\|^2_{2.5,+} + \|w\|^2_{2.5,+}\Big]\,,\\ \|q\|^2_{2.5,-} &\le C\Big[\|\bar w\|^2_{2.5,-} + \|w_t\|^2_{1.5,+} + \|w\|^2_{1.5,+}\Big]\,. \end{aligned}$$

Combining these two estimates and (5.11), by interpolations we find that

$$\|q\|^2_{\mathcal{V}_-^{3.5}(T)} \le N(u_0) + C_{\kappa,\delta}\sqrt{T} + \delta\|\bar q\|^2_{2,-} + CT\Big[\|\bar w\|^2_{\mathcal{V}_-^{4.5}(T)} + \|\bar w_t\|^2_{\mathcal{V}_-^{2.5}(T)}\Big]\,. \tag{5.15}$$

For the regularity of v, we mimic the divergence and curl estimates as in [4]. In Ω^-,

$$(\varepsilon_{ijk}\bar{A}_j^\ell v_{,\ell}^{-k})_t + \bar{A}_r^s(\bar{v}^{-r} - \bar{v}_e^{-r})(\varepsilon_{ijk}\bar{A}_j^\ell v_{,\ell}^{-k})_{,s} = B^i(v) \tag{5.16}$$

where

$$\begin{aligned} B^i(v) &= \varepsilon_{ijk}\Big[\bar{A}_r^\ell \bar{v}_{\kappa,s}^{-r}\bar{A}_j^s v_{,\ell}^{-k} + \bar{A}_r^s(\bar{v}^{-r} - \bar{v}_e^{-r})\bar{A}_{j,s}^\ell v_{,\ell}^{-k} - \bar{A}_j^\ell[\bar{A}_r^s(\bar{v}^{-r} - \bar{v}_e^{-r})]_{,\ell} v_{,s}^{-k}\Big] \\ &= \varepsilon_{ijk}\Big[\bar{A}_r^\ell \bar{v}_{\kappa,s}^{-r}\bar{A}_j^s v_{,\ell}^{-k} - \bar{A}_j^\ell \bar{A}_r^s(\bar{v}_{,\ell}^{-r} - \bar{v}_{e,\ell}^{-r})v_{,s}^{-k}\Big], \end{aligned}$$

a function of ∇v, $\nabla \bar{v}$ and $\nabla \bar{\eta}$, where we use the identity $\bar{A}_r^s\bar{A}_{j,s}^\ell = \bar{A}_j^s\bar{A}_{r,s}^\ell$. Let $\bar{\zeta}$ be the solution to

$$\bar{\zeta}_t^i = [\bar{A}_j^i(\bar{v}^{-j} - \bar{v}_e^{-j})] \circ \bar{\zeta},$$

i.e., $\bar{\zeta}$ is the flow map of the velocity field $\bar{A}^T(\bar{v} - \bar{v}_e)$, then

$$\varepsilon_{ijk}\bar{A}_j^\ell v_{,\ell}^{-k} = \Big[\operatorname{curl} u_0 + \int_0^\cdot B^i(v)\circ\bar{\zeta}ds\Big]\circ\bar{\zeta}^{-1}. \tag{5.17}$$

Since

$$\Big[\int_0^t K(\bar{\zeta}(y,s),s)ds\Big]\circ\bar{\zeta}^{-1}(x,t) = \int_0^t K(\bar{\zeta}(x,s-t),s)ds,$$

(5.17) implies

$$\varepsilon_{ijk}\bar{A}_j^\ell v_{,\ell}^{-k}(x,t) = (\operatorname{curl} u_0)\circ\bar{\zeta}^{-1}(x,t) + \int_0^t B^i(v)\circ\bar{\zeta}(x,s-t)ds. \tag{5.18}$$

We use (5.18) as the fundamental equality to proceed to vorticity estimates in Ω^-. Since $\|\bar{\zeta}(t)\|_{4.5,-}^2 \le M_0 + CT\|\bar{v}^-\|_{\mathcal{V}_-^{4.5}(T)}^2 \equiv C(\bar{v}^-)$, (5.18) implies that

$$\|\operatorname{curl}_{\bar{\eta}_\kappa} v\|_{3.5,-}^2 \le C(\bar{v}^-)\Big[N(u_0) + \int_0^t \|v\|_{4.5,-}^2 ds\Big]. \tag{5.19}$$

Transforming back to the domain $\bar{\eta}_\kappa(\Omega^-)$, we find that

$$\|\operatorname{curl} u\|_{H^{3.5}(\bar{\eta}_\kappa(\Omega^-))}^2 \le C(\bar{v}^-)\Big[N(u_0) + \int_0^t \|u\|_{H^{3.5}(\bar{\eta}_\kappa(\Omega^-))}^2 ds\Big].$$

We remark here that the restriction of obtaining higher regularity is mainly due to the presence of $\nabla\bar{A}$ in $B(v)$ that comes from the transport term. Boundary conditions (5.2c) and (5.2d) imply

$$\|u\cdot N\|_{H^4(\partial\bar{\eta}_\kappa(\Omega^-))}^2 = \|u\cdot N\|_{H^4(\bar{\eta}_\kappa(\Gamma))}^2 + \|u\cdot N\|_{H^4(\bar{\eta}_\kappa(\partial\Omega))}^2 \le C\|w\|_{4.5,+}^2.$$

These two estimates and the divergence free constraint $\operatorname{div} u = 0$ lead to

$$X(T) \le C\|w\|_{\mathcal{V}_+^{4.5}(T)}^2 + C(\bar{v}^-)\Big[TN(u_0) + \int_0^T X(t)dt\Big],$$

where $X(T) = \displaystyle\int_0^T \|u\|_{H^{4.5}(\bar{\eta}_\kappa(\Omega^-))}^2 dt$. Therefore, the Gronwall inequality implies

$$\int_0^t \|u\|_{H^{4.5}(\bar{\eta}_\kappa(\Omega^-))}^2 ds \le [1 + C(\bar{v}^-)T]N(u_0) + C_{\kappa,\delta}\sqrt{T} + \delta\|\bar{q}\|_{2,-}^2$$

or equivalently,

$$\int_0^T \|v\|_{4.5,-}^2 dt \le [1 + C(\bar{v}^-)T]N(u_0) + C_{\kappa,\delta}\sqrt{T} + \delta\|\bar{q}\|_{2,-}^2.$$

For T even smaller (so that $C(\bar v^-)T$ is small), it follows from (5.2a) that

$$\int_0^T \|v_t\|^2_{2.5,-}dt \le C\int_0^T \Big[\|v\|^2_{3.5,-} + \|q\|^2_{3.5,-}\Big]ds$$
$$\le N(u_0) + C_{\kappa,\delta}\sqrt{T} + \delta\|\bar q\|^2_{2,-} + CT\Big[\|\bar w\|^2_{\mathcal{V}^{4.5}_-(T)} + \|\bar w_t\|^2_{\mathcal{V}^{2.5}_-(T)}\Big]\,.$$

Therefore,

$$\|v\|^2_{\mathcal{V}^{4.5}_-(T)} + \|v_t\|^2_{\mathcal{V}^{2.5}_-(T)} + \|q\|^2_{\mathcal{V}^{3.5}_-(T)}$$
$$\le N(u_0) + C_{\kappa,\delta}\sqrt{T} + \delta\|\bar q\|^2_{2,-} + CT\Big[\|\bar w\|^2_{\mathcal{V}^{3.5}_-(T)} + \|\bar w_t\|^2_{\mathcal{V}^{2.5}_-(T)}\Big]\,. \tag{5.20}$$

In the following sections, we will always assume that the initial input $\bar q$ satisfies $\|\bar q\|^2_{\mathcal{V}^{3.5}_-(T)} \le N(u_0)+1$. We can choose a fixed but positive $\delta < \dfrac{1}{2(N(u_0)+1)}$, and let L be the collection of those elements $v \in L^2(0,T;H^{4.5}(\Omega^-))$ so that

$$\|v\|^2_{\mathcal{V}^{4.5}_-(T)} + \|v_t\|^2_{\mathcal{V}^{2.5}_-(T)} \le N(u_0)+1\,.$$

For a fixed $\kappa > 0$, we choose T small enough so that

$$C_{\kappa,\delta}\sqrt{T} + CT\Big[N(u_0)+1\Big] \le \frac{1}{2}\,.$$

Clearly the map Ψ maps from L into L. Similar to the proof in Section 5.4, Ψ can be shown to be weakly continuous in $L^2(0,T_\kappa;H^{5.5}(\Omega^-))$. Since L is a closed convex set in $L^2(0,T;H^{4.5}(\Omega^-)))$, by Tychonoff fixed-point theorem, there is a fixed-point v of the map Ψ which provides a solution to (5.2). Uniqueness follows from the fact that (5.2) is linear.

Remark 2. *It follows from (5.20) that*

$$\|v\|^2_{\mathcal{V}^{4.5}_-(T)} + \|v_t\|^2_{\mathcal{V}^{2.5}_-(T)} + \|q\|^2_{\mathcal{V}^{3.5}_-(T)} \le N(u_0)+1\,. \tag{5.21}$$

5.4. **Weak continuity of the map Φ.** Let $(\bar v^\pm_m, \bar q_m)$ converges weakly to $(\bar v^\pm, \bar q)$ in the space $L^2(0,T;H^{5.5}(\Omega^\pm))\times L^2(0,T;H^{3.5}(\Omega^-)/\mathbb{R})$, $\Phi(\bar v^+_m, \bar v^-_m, \bar q_m) = (w_m, v_m, q_m)$ and $\Phi(\bar v^+, \bar v^-, \bar q) = (w, v, q)$. Suppose that $\mathcal{J}^\pm_{\kappa m}$, $\bar a^\pm_{\kappa m}$, $\bar n^\pm_{\kappa m}$ are constructed from $\bar v^\pm_\kappa$ accordingly. By the property of convolution by layers and the weak convergence, we have $(\mathcal{J}^\pm_{\kappa m}, \bar a^\pm_{\kappa m}, \bar n^\pm_{\kappa m})$ converges to $(\bar{\mathcal{J}}^\pm_\kappa, \bar a^\pm_\kappa, \bar n^\pm_\kappa)$ strongly in $[L^\infty(0,T;H^{4.5}(\Omega^\pm))]^3$. Since (w_m, r_m) satisfies

$$\int_{\Omega^+}\rho^+\mathcal{J}^+_{\kappa m}w^i_{mt}\varphi^i dx - \int_{\Omega^+} r_m(\bar a^+_{\kappa m})^j_i\varphi^i_{,j}dx + \kappa\int_\Gamma g_0^{\alpha\beta}(w_m\cdot\bar n^+_{\kappa m})_{,\alpha}(\varphi\cdot\bar n^+_{\kappa m})_{,\beta}dS$$
$$= \sigma\int_\Gamma\Big[\bar q_m + L_{\bar g_{\kappa m}}(\bar\eta^+_m)\cdot\bar n^+_{\kappa m}\Big](\bar a^+_{\kappa m})^j_i N_j\varphi^i dS \qquad \forall\varphi\in H^{\frac32}(\Omega^+)\,,$$

and (w_m, r_m) are uniformly bounded in $L^2(0,T;H^{5.5}(\Omega^+))\times L^2(0,T;H^{3.5}(\Omega^+))$, it follows that there exists $(\tilde w, \tilde r)$ so that

$$\int_{\Omega^+}\rho^+\bar{\mathcal{J}}^+_\kappa\tilde w^i_t\varphi^i dx - \int_{\Omega^+}\tilde r(\bar a^+_\kappa)^j_i\varphi^i_{,j}dx + \kappa\int_\Gamma g_0^{\alpha\beta}(\tilde w\cdot\bar n^+_\kappa)_{,\alpha}(\varphi\cdot\bar n^+_\kappa)_{,\beta}dS$$
$$= \sigma\int_\Gamma\Big[\bar q + L_{\bar g}(\bar\eta^+)\cdot\bar n^+_\kappa\Big](\bar a^+_\kappa)^j_i N_j\varphi^i dS \qquad \forall\varphi\in H^{\frac32}(\Omega^+)\,.$$

By the uniqueness of the solution to the linearized problem, $\tilde w = w$. Similar argument shows that the solution (v_m, q_m) to problem (5.2) with all the fixed coefficients constructed from $\bar v_m$ converges weakly to (v,q), the solution to problem (5.2). Therefore, the weak continuity of the map Φ is established.

5.5. **The fixed-point argument.** The only thing we need to check is that if there is $T>0$ and a closed convex set $K\subseteq L^2(0,T;H^{5.5}(\Omega^+))\times L^2(0,T;H^{4.5}(\Omega^-)/\mathbb{R})\times L^2(0,T;H^{3.5}(\Omega^-))$ so that Φ maps from K into K. Let K be defined as the collection of those elements $(w^\pm,q)\in L^2(0,T;H^{5.5}(\Omega^+))\times L^2(0,T;H^{4.5}(\Omega^-))\times L^2(0,T;H^{3.5}(\Omega^-)/\mathbb{R})$ so that

$$\|w^+\|^2_{\mathcal{V}_+^{5.5}(T)}+\|w^+_t\|^2_{\mathcal{V}_+^{2.5}(T)}\le N(u_0)+1\,,$$
$$\|w^-\|^2_{\mathcal{V}_-^{4.5}(T)}+\|w^-_t\|^2_{\mathcal{V}_-^{2.5}(T)}+\|q\|^2_{\mathcal{V}_-^{3.5}(T)}\le N(u_0)+1\,.$$

Recall that δ is fixed from the previous section. Similar to the proof in the previous section, we choose $T>0$ small enough so that

$$C_{\kappa,\delta}\sqrt{T}+T(N(u_0)+1)\le\frac{1}{2}\,.$$

Then by estimates (5.11) and (5.21), the map Φ indeed maps from K into K. Therefore, the Tychonoff fixed-point theorem implies the existence of a fixed-point (v,q) of Φ.

Remark 3. *This T is κ-dependent.*

Remark 4. *Once a solution to problem (4.1) is obtained, without loss of generality, we may assume that the pressure and its time derivatives satisfy the Poincaré inequality (5.22): let $\bar q=\frac{1}{|\Omega|}\Big(\int_{\Omega^+}q^+dx+\int_{\Omega^-}q^-dx\Big)$. Since q^+ and q^- is uniquely determined up to the addition of a constant (constant in space), we can replace q^+ and q^- by $q^+-\bar q(\equiv Q^+)$ and $q^--\bar q(\equiv Q^-)$*

$$\|Q\|^2_{0,\pm}=\|Q\|^2_0\le C\|\nabla Q\|^2_0=C\|\nabla Q\|^2_{0,\pm}\,. \tag{5.22}$$

5.6. **Estimates of the divergence and curl of the velocity field.**

5.6.1. *Divergence and curl estimates.* In Ω^+, we can apply exactly the same technique as in [4] to conclude the following lemma.

Lemma 5.1 (Divergence and curl estimates in Ω^+). *Let $L_1=\operatorname{curl}$ and $L_2=\operatorname{div}$, and let $\eta_0:=\eta(0)$ and*

$$M_0^+:=P(\|u_0^+\|_{2.5+\mathfrak{n},+},|\Gamma|_{3.5+\mathfrak{n}},\sqrt{\kappa}\|u_0^+\|_{1.5+3\mathfrak{n},+},\sqrt{\kappa}|\Gamma|_{1+3\mathfrak{n}})$$

denote a polynomial function of its arguments. Then for $j=1,2$,

$$\begin{aligned}&\sup_{t\in[0,T]}\|\sqrt{\kappa}L_j\eta^+(t)\|^2_{2.5+\mathfrak{n},+}+\sum_{k=0}^{\mathfrak{n}+1}\sup_{t\in[0,T]}\|L_j\partial_t^k\eta^+(t)\|^2_{1.5+\mathfrak{n}-k,+}\\&+\sum_{k=0}^{\mathfrak{n}+1}\int_0^T\|\sqrt{\kappa}L_j\partial_t^kv^+\|^2_{2.5+\mathfrak{n}-k,+}dt\le M_0^++CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\,.\end{aligned} \tag{5.23}$$

Similar to the way of obtaining (5.19), the following lemma is valid as well.

Lemma 5.2 (Divergence and curl estimates in Ω^-). *Let $\mathfrak{n}$, L_1 and L_2 be defined as those in Lemma 5.1, and*

$$M_0^-:=P(\|u_0^-\|_{2.5+\mathfrak{n},-},|\Gamma|_{3.5+\mathfrak{n}},\sqrt{\kappa}\|u_0^-\|_{1.5+3\mathfrak{n},-},\sqrt{\kappa}|\Gamma|_{1+3\mathfrak{n}})\,.$$

Then for $j=1,2$,

$$\sum_{k=1}^{\mathfrak{n}+2}\sup_{t\in[0,T]}\|L_j\partial_t^kv^-(t)\|^2_{1.5+\mathfrak{n}-k,-}\le M_0^-+CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\,. \tag{5.24}$$

5.6.2. *$H^{-0.5}$-estimates for $v^{\pm}_{ttt}$ on the boundary Γ and $\partial\Omega$.* By (4.1b),

$$
\begin{aligned}
(\operatorname{curl} v^{\pm}_{ttt})^i &= \varepsilon_{ijk}\Big[[\delta^\ell_j - (a_\kappa)^\ell_j]v^{\pm k}_{ttt,\ell} - (a_{\kappa tt})^\ell_j v^{\pm k}_{t,\ell} - (a_{\kappa t})^\ell_j v^{\pm k}_{tt,\ell} && \text{in } \Omega^{\pm}\,,\\
&\qquad - \Big((a_\kappa)^\ell_j [A^s_r(v^{-r} - v^{-r}_e)v^{-k}_{,s}]_{,\ell}\Big)_{tt}\Big]\\
\operatorname{div} v^{\pm}_{ttt} &= (\delta^\ell_j - A^\ell_j)v^{\pm i}_{ttt,\ell} - (A_{ttt})^j_i v^{\pm i}_{,j} - 3(A_{tt})^j_i v^{\pm i}_{t,j} - 3(A_t)^j_i v^{\pm i}_{tt,j} && \text{in } \Omega^{\pm}\,.
\end{aligned}
$$

Since $v^{\pm}_{ttt} \in L^2(0,T;H^{1.5}(\Omega^{\pm}))$ (with κ dependent estimate), $\operatorname{curl} v^{\pm}_{ttt}$ and $\operatorname{div} v^{\pm}_{ttt}$ are both in $L^2(\Omega^{\pm})$ and hence by (3.3), $\|v^{\pm}_{ttt}\|_{H^{-0.5}(\partial\Omega^{\pm})}$ exists. For $\varphi \in H^1(\Omega^-)$,

$$
\begin{aligned}
\int_{\Omega^-} \operatorname{curl} v^-_{ttt}\cdot\varphi dx &\le \varepsilon_{ijk}\int_{\partial\Omega^-}[\delta^\ell_j - (a_\kappa)^\ell_j]v^k_{ttt}\varphi^i N_\ell dS\\
&\quad - \varepsilon_{ijk}\int_{\partial\Omega^-} A^\ell_j (a_\kappa)^s_r (v^{-r} - v^{-r}_e)v^{-k}_{tt,\ell}\varphi^i N_s dS + C\mathcal{P}(E_\kappa(t))\|\varphi\|_{1,-}\,.
\end{aligned}
$$

Since $a_\kappa = \mathrm{Id}$ and $v^-_e = 0$ outside Ω', we find that

$$
\begin{aligned}
&\varepsilon_{ijk}\int_{\partial\Omega^-} A^\ell_j (a_\kappa)^s_r (v^{-r} - v^{-r}_e)v^{-k}_{tt,\ell}\varphi^i N_s dS\\
&= \varepsilon_{ijk}\Big[\int_\Gamma A^\ell_j\sqrt{g_\kappa}[n_\kappa\cdot(v^-_e - v^-)]v^{-k}_{tt,\ell}\varphi^i dS + \int_{\partial\Omega}(v^-\cdot N)v^{-k}_{tt,j}\varphi^i dS\Big]\\
&= 0\,,
\end{aligned}
$$

where we use the boundary condition (4.1e) and (4.1f) with $v^-_e = v^+$ on Γ to conclude the last equality. Therefore,

$$
\begin{aligned}
\Big|\int_{\Omega^-}\operatorname{curl} v^-_{ttt}\cdot\varphi dx\Big| &\le \varepsilon_{ijk}\int_\Gamma[\delta^\ell_j - (a_\kappa)^\ell_j]v^k_{ttt}\varphi^i dS + C\mathcal{P}(E_\kappa(t))\|\varphi\|_{1,-}\\
&\le C\Big[t|v_{ttt}|_{-0.5,\pm} + \mathcal{P}(E_\kappa(t))\Big]\|\varphi\|_{1,-}
\end{aligned}
$$

which implies

$$
\|\operatorname{curl} v^-_{ttt}\|_{H^1(\Omega^-)'} \le C\Big[t|v^-_{ttt}|_{-0.5,\pm} + \mathcal{P}(E_\kappa(t))\Big]\,.
$$

Similarly,

$$
\begin{aligned}
\|\operatorname{curl} v^+_{ttt}\|_{H^1(\Omega^+)'} &\le C\Big[t|v^+_{ttt}|_{-0.5,\pm} + \mathcal{P}(E_\kappa(t))\Big]\,,\\
\|\operatorname{div} v^{\pm}_{ttt}\|_{H^1(\Omega^{\pm})'} &\le C\Big[t\|v_{ttt}\|_{H^{-0.5}(\partial\Omega^{\pm})} + \mathcal{P}(E_\kappa(t))\Big]\,.
\end{aligned}
$$

Therefore, by (3.3),

$$
\begin{aligned}
\|v^{\pm}_{ttt}\|_{H^{-0.5}(\partial\Omega^{\pm})} &\le C\Big[\|v^{\pm}_{ttt}\|_{L^2(\Omega)} + \|\operatorname{div} v^{\pm}_{ttt}\|_{H^1(\Omega)'} + \|\operatorname{curl} v^{\pm}_{ttt}\|_{H^1(\Omega)'}\Big]\\
&\le CT\|v^{\pm}_{ttt}\|_{H^{-0.5}(\partial\Omega^{\pm})} + C\mathcal{P}(E_\kappa(t))\,.
\end{aligned}
$$

It then follows from choosing $T > 0$ small enough that

$$
|v_{ttt}|_{-0.5,\pm} + |v^-_{ttt}|_{-0.5,\partial\Omega} \le C\mathcal{P}(E_\kappa(t))\,. \tag{5.25}
$$

6. Estimates for velocity, pressure, and their time derivatives at time $t = 0$

In this section, we estimate the time derivatives of the velocity and pressure at the initial time $t = 0$. We use w_k, $k = 1, 2, 3$, and q_ℓ, $\ell = 0, 1, 2$, to denote $\partial_t^k v(0)$ and $\partial_t^\ell q(0)$.Let φ_κ be defined by

$$
\begin{aligned}
[\mathcal{J}^{-1} a_i^j (\mathcal{J}^{-1} a_i^k \varphi_\kappa)_{,k}]_{,j} &= 0 && \text{in} \quad \Omega^+ \,, && (6.1a)\\
\varphi_\kappa &= -\sigma L_g(\eta_e)\cdot n_\kappa - \kappa \Delta_0 (v\cdot n_k) && \text{on} \quad \Gamma \,, && (6.1b)\\
\varphi_\kappa &= 0 && \text{on} \quad \partial\Omega \,, && (6.1c)
\end{aligned}
$$

and the quantities φ_0, φ_1 and φ_2 be defined by $\varphi_\kappa(0)$, $\varphi_{\kappa t}(0)$ and $\varphi_{\kappa tt}(0)$, respectively.

Let q_0^+ and q_0^- denote the initial pressure $q(0)$ in Ω^+ and Ω^-, respectively, then q_0^+ and q_0^- satisfy

$$
\begin{aligned}
-\frac{1}{\rho^+}\Delta(q_0^+ - \varphi_0) &= f^+ && \text{in} \quad \Omega^+ \,, && (6.2a)\\
-\frac{1}{\rho^-}\Delta q_0^- &= f^- && \text{in} \quad \Omega^- \,, && (6.2b)\\
\frac{1}{\rho^+}\frac{\partial q_0^+}{\partial N} &= -w_1^+ \cdot N && \text{on} \quad \partial\Omega^+ \,, && (6.2c)\\
\frac{1}{\rho^-}\frac{\partial q_0^-}{\partial N} &= (-w_1^- + \nabla_{(v_e^-(0) - u_0^-)} u_0^-)\cdot N && \text{on} \quad \partial\Omega^- \,, && (6.2d)
\end{aligned}
$$

where $f^\pm = (\nabla u_0^\pm)^T : \nabla u_0^\pm$, and N denotes the unit normal of Γ from Ω^- into Ω^+, or the outward unit normal of $\partial\Omega$.

Remark 5. *The right-hand side of (6.2b) is in fact* $f^- - (v_e^-(0) - u_0^-)\cdot \nabla \operatorname{div} u_0^-$ *while the last term is zero by the divergence free constraint of the initial data.*

For all $\psi \in H^1(\Omega^+) \cap H^1(\Omega^-)$ so that $\psi^+ = \psi^-$ on Γ, we have

$$
\begin{aligned}
&\frac{1}{\rho^+}\int_{\Omega^+} \nabla(q_0^+ - \varphi_0)\cdot\nabla\psi dx + \frac{1}{\rho^-}\int_{\Omega^-}\nabla q_0^- \cdot \nabla\psi dx = \int_{\Omega^+} f^+\psi dx + \int_{\Omega^-} f^-\psi dx\\
&-\int_{\partial\Omega}(w_1^-\cdot N - \nabla_{v_e^-(0)-u_0^-} u_0^-)\psi dS - \int_\Gamma (w_1^+ - w_1^- + \nabla\varphi_0)\cdot N\psi dS \,. \qquad (6.3)
\end{aligned}
$$

Since $[v\cdot n_\kappa]_\pm = 0$ and $v^+\cdot n_\kappa = 0$ on $\partial\Omega$, it follows that

$$
\begin{aligned}
w_1^+\cdot N + u_0^+\cdot n_{\kappa t}(0) &= w_1^-\cdot N + u_0^-\cdot n_{\kappa t}(0) && \text{on} \quad \Gamma \,,\\
w_1^-\cdot N &= 0 && \text{on} \quad \partial\Omega \,,
\end{aligned}
$$

and hence (6.3) implies

$$
\begin{aligned}
&\frac{1}{\rho^+}\int_{\Omega^+} \nabla q_0^+ \cdot\nabla\psi dx + \frac{1}{\rho^-}\int_{\Omega^-}\nabla (q_0^- - \varphi_0) \cdot \nabla\psi dx = \int_{\Omega^+} f^+\psi dx + \int_{\Omega^-} f^-\psi dx\\
&+\int_\Gamma [(u_0^+ - u_0^-)\cdot n_{\kappa t}(0) + \nabla\varphi_0\cdot N]\psi dS + \int_{\partial\Omega} \nabla_{v_e^-(0)-u_0^-} u_0^-\cdot N\psi dS \,. \qquad (6.4)
\end{aligned}
$$

Let $Q_0 = q_0^+ - \varphi_0$ in Ω^+ and $Q_0 = q_0^-$ in Ω^-. Since $Q_0^+ = Q_0^-$ on Γ, we can use Q_0 (and its difference quotients) as a test function in (6.4). Since $n_{\kappa t}(0) =$

$-g_0^{\alpha\beta}(u_{0\kappa,\beta}\cdot N)\mathrm{Id}_{,\alpha}$ and $\|v_e(0)\|_{k,-}\le C\|u_0\|_{k,+}$, by standard difference quotient technique, for $s>1.5$ for $\mathfrak{n}=2$ or $s>1.75$ for $\mathfrak{n}=3$,

$$\begin{aligned}
&\|q_0^+\|^2_{s,+}+\|q_0^- -\varphi_0\|^2_{s,-}\\
&\le C\|f\|^2_{s-2,\pm}+C|u_0^+\cdot\bar n_{\kappa t}(0)-u_0^-\cdot\bar n_{\kappa t}(0)+\nabla_{v_e^-(0)-u_0^-}u_0^-\cdot N+\nabla\varphi_0\cdot N|^2_{s-1.5}\\
&\le C\mathcal{P}(\|u_0\|^2_{s,\pm})+C(|\Gamma|^2_{s+1.5}+\kappa|u_0\cdot N|^2_{s+1.5})\,.
\end{aligned}$$

By the elliptic estimate for φ in (6.1) together with (4.1b), we find that for $s>1+s(\mathfrak{n})$,

$$\|w_1\|^2_{s-1,\pm}+\|q_0\|^2_{s,\pm}\le C\mathcal{P}(\|u_0\|^2_{s,\pm},|\Gamma|^2_{s+1.5},\|\sqrt{\kappa}u_0^+\|^2_{s+2,+})\,. \tag{6.5}$$

For $j=1,2$, the quantities $q_1^\pm$ and $q_2^\pm$ satisfy

$$\begin{aligned}
\frac{1}{\rho^\pm}\Delta(q_j^\pm-\varphi_j^+)&=h_j^\pm+k_j^-+\phi_j^+ &&\text{in}\quad\Omega^\pm\,,\\
\frac{1}{\rho^\pm}\frac{\partial q_j^\pm}{\partial N}&=-(\partial_t^{j+1}v^\pm)(0)\cdot N+j(\nabla q_{j-1}^\pm)^T\nabla u_{0\kappa}^\pm N &&\text{on}\quad\partial\Omega^\pm\,,\\
&\quad+2(j-1)\nabla q_0^\pm(\nabla u_{0\kappa}^\pm\nabla u_{0\kappa}^\pm-\nabla w_{1\kappa}^\pm)N+B_j^-
\end{aligned}$$

where

$$\begin{aligned}
h_1^\pm&=-2u^j_{0\kappa,i}w^{\pm i}_{1,j}+2u^r_{0\kappa,i}u^\ell_{0\kappa,r}u^{\pm i}_{0,\ell}-w^\ell_{1\kappa,i}u^{\pm i}_{0,\ell}+\nabla q_0^\pm\cdot\Delta u_{0\kappa}+u^i_{0\kappa,j}q^\pm_{0,ij}\,,\\
h_2^\pm&=-3u^j_{0\kappa,i}w^{\pm i}_{2,j}+6u^r_{0\kappa,i}u^\ell_{0\kappa,r}w^{\pm i}_{1,\ell}-3w^\ell_{1\kappa,i}w^{\pm i}_{1,\ell}+6[\nabla u_{0\kappa}\nabla u_{0\kappa}\nabla u_{0\kappa}]^T:\nabla u_0^\pm\\
&\quad-4(\nabla u_0^\pm\nabla w_{1\kappa}):(\nabla u_{0\kappa})^T-2(\nabla u_0^\pm\nabla u_{0\kappa}):(\nabla w_{1\kappa})^T+(\nabla w_{2\kappa})^T:\nabla u_0^\pm\\
&\quad+\mathrm{div}[2(\nabla w_{1\kappa})^T\nabla q_1^\pm+2(\nabla u_{0\kappa}\nabla u_{0\kappa})^T\nabla q_0^\pm-(\nabla w_{1\kappa})^T\nabla q_0^\pm]\,,\\
k_1&=\Big[-u^k_{0\kappa,i}(v_e(0)^{-j}-u_0^{-j})u^{-i}_{0,k}+(v^{-j}_{et}(0)-w_1^{-j})u^{-i}_{0,j}+(v_e^{-j}(0)-u_0^-)w^{-i}_{1,j}\Big]_{,i}\,,\\
k_2&=\Big[(2u^\ell_{0\kappa,j}u^k_{0\kappa,\ell}-w^k_{1\kappa,j})(v_e^{-j}(0)-u_0^{-j})u^{-i}_{0,k}+(v^{-j}_{ett}(0)-w_2^{-j})u^{-i}_{0,j}\\
&\quad+(v_e^{-j}(0)-u_0^{-j})w^{-i}_{2,j}-2u^k_{0\kappa,j}(v^{-j}_{et}(0)-w_1^{-j})w^{-i}_{1,k}-2u^k_{0\kappa,j}(v_e^{-j}(0)-u_0^{-j})w^{-i}_{1,k}\\
&\quad+2(v^j_{et}(0)-w_1^{-j})w^{-i}_{1,j}\Big]_{,i}\,,
\end{aligned}$$

and

$$\begin{aligned}
\phi_1&=-2\nabla u_{0\kappa}:\nabla^2\varphi_0-\nabla\varphi_0\cdot\Delta u_{0\kappa}\,,\\
\phi_2&=-4(\nabla u_{0\kappa}\nabla u_{0\kappa}):\nabla^2\varphi_0+2\nabla w_{1\kappa}:\nabla^2\varphi_0-\mathrm{div}(\nabla u_{0\kappa}\nabla u_{0\kappa})\cdot\nabla\varphi_0\\
&\quad+\Delta w_{1\kappa}\cdot\nabla\varphi_0+2\Delta u_{0\kappa}\cdot\nabla\varphi_1+4\nabla u_{0\kappa}:\nabla^2\varphi_1+2\nabla[(\nabla u_{0\kappa})^T\nabla\varphi_1]:\nabla u_{0\kappa}\,,\\
B_1&=(v_e^-(0)-u_0^-)^T\nabla u_{0\kappa}(\nabla u_0^-)^TN-(v_{et}^-(0)-w_1^-)^T\nabla u_0^-N-(v_e^-(0)-u_0^-)^T\nabla w_1^-N\,,\\
B_2&=\Big[\nabla u_0^-(2\nabla u_{0\kappa}\nabla u_{0\kappa}-\nabla w_{1\kappa})(v_e^-(0)-u_0^-)+\nabla u_0^-(v_{ett}^-(0)-w_2^-)\\
&\quad+\nabla w_2^-(v_e^-(0)-u_0^-)-2(\nabla u_0^-\nabla u_{0\kappa})(v_{et}^-(0)-w_1^-)-2\nabla w_1^-\nabla u_{0\kappa}(v_e^-(0)-u_0^-)\\
&\quad-2\nabla w_1^-(v_{et}^-(0)-w_1^-)\Big]N\,,
\end{aligned}$$

where $u_{0\kappa}$, $w_{1\kappa}$ and $w_{2\kappa}$ are $M^+u^+_{0\kappa}$, $M^+w^+_{1\kappa}$ and $M^+w^+_{2\kappa}$, respectively. Similar to the estimate of $q_0^{\pm}$, since on Γ,

$$
\begin{aligned}
q_j^+ - \varphi_j = q_j^- \qquad &\text{for } j=1,2\,,\\
[v_{tt}^+(0) - v_{tt}^-(0)]\cdot N &= -2[w_1]_\pm \cdot \bar{n}_{\kappa t}(0) - [u_0]_\pm \cdot \bar{n}_{\kappa tt}(0)\,,\\
[v_{ttt}^+(0) - v_{ttt}^-(0)]\cdot N &= -3[w_2]_\pm \cdot \bar{n}_{\kappa t}(0) - 3[w_1]_\pm \cdot \bar{n}_{\kappa tt}(0) - [u_0]_\pm \cdot \bar{n}_{\kappa ttt}(0)\,,
\end{aligned}
$$

and on $\partial\Omega$,

$$v_{tt}^-(0)\cdot N = v_{ttt}^-(0)\cdot N = 0\,,$$

we find that for $s \geq 2$,

$$
\begin{aligned}
&\|w_2\|^2_{s-2,\pm} + \|q_1\|^2_{s-1,\pm}\\
&\leq C\mathcal{P}(\|u_0\|^2_{s,\pm}, |\Gamma|^2_{s+1.5}, \|\sqrt{\kappa}u_0^+\|^2_{s+2,+}, \|\sqrt{\kappa}w_1^+\|^2_{s+1,+})
\end{aligned}
$$

and for $s \geq 3$,

$$
\begin{aligned}
&\|w_3\|^2_{s-3,\pm} + \|q_2\|^2_{s-2,\pm}\\
&\leq C\mathcal{P}(\|u_0\|^2_{s,\pm}, |\Gamma|^2_{s+1.5} + \kappa|u_0\cdot N|^2_{s+1.5} + \kappa|w_1^+\cdot N|^2_{s+0.5} + \kappa|w_2^+\cdot N|^2_{s-0.5})\,,
\end{aligned}
$$

where we also use the boundedness of the extension operator M so that

$$\|\partial_t^k v_e(0)\|_{s,-} \leq C\|w_k^+\|^2_{s,+}\,.$$

7. Pressure estimates

The estimates for the pressure and its time derivatives are exactly the same as (12.1) in [4]. In [4], the L^2-estimate for the pressure is found by studying a Dirichlet problem, but in the two-phase problem with fixed outer boundary, the L^2-estimate is not necessary because of the Poincare inequality. Therefore,

$$\|q(t)\|^2_{3.5,\pm} + \|q_t(t)\|^2_{2.5,\pm} + \|q_{tt}(t)\|^2_{1,\pm} \leq C\mathcal{P}(E_\kappa(t)) \tag{7.1}$$

for some constant C independent of κ.

Remark 6. *The estimates for q^-, q_t^- and q_{tt}^- require the control of $\|v^-\|_{3.5,-}$, $\|v_t^-\|_{2.5,-}$, $\|v_{tt}^-\|_{1.5,-}$ and $\|v_{ttt}^-\|_{0,-}$, respectively. This is the only reason we need to include the estimates for $\partial_t^j v^-$ into our definition of energy (2.1). Note that we do not need $\|\eta^-\|_{4.5,-}$ in order to control $\partial_t^j q^-$.*

8. κ-independent estimates

We also make use of the following inequality which follows from Morrey's inequality (see (2.6) in [4]). For $U \in W^{1,p}(\Gamma)$,

$$|U_\kappa(x) - U(x)| \leq C\kappa^{1-\frac{n}{p}}|DU|_p \tag{8.1}$$

Test (4.1) against a function $\varphi \in H^{\frac{3}{2}}(\Omega^+)\cap H^{\frac{3}{2}}(\Omega^-)$ with $\varphi^-\cdot N = 0$ on $\partial\Omega$,

$$
\begin{aligned}
&\int_{\Omega^+} \rho^+ \mathcal{J}_\kappa v_t^{+i}\varphi^i dx + \int_{\Omega^-}\rho^-\Big[\mathcal{J}_\kappa v_t^{-i} + (a_\kappa)^k_j(v^{-j} - v_e^{-j})v^{-i}_{,k}\Big]\varphi^i dx\\
&+ \int_{\Omega^+}(a_\kappa)^j_i q^+_{,j}\varphi^{+i}dx + \int_{\Omega^-}(a_\kappa)^j_i q^-_{,j}\varphi^{-i}dx = 0\,.
\end{aligned} \tag{8.2}
$$

Similar to those estimates in [4], the κ-independent estimate consists of studying the three time differentiated problem, three tangential space differentiated problem, and the intermediate problems with mixing time and tangential space derivatives.

Most of the estimates are essentially the same as those in [4], and in the following sections we only list those terms which required further study.

Before proceeding, we remark that those energy estimates in [4] can be refined a bit further. For example, the energy estimate for the third time-differentiated κ-problem ((12.6) in [4]) can be refined as

$$\sup_{t\in[0,T]}\Big[\|v_{ttt}\|_0^2+|v_{tt}\cdot n_\kappa|_1^2\Big]+\int_0^T|\sqrt{\kappa}\partial_t^3 v\cdot n_\kappa|_1^2dt\le M_0(\delta)+\delta\sup_{t\in[0,T]}E_\kappa(t)$$
$$+CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+C(\delta)\Big[\|v_t\|_{2.5}^2+\|v\|_{3.5}^2+\|\eta_e\|_{4.5}^2+\int_0^T\|\sqrt{\kappa}v_{tt}\|_{2.5}^2dt\Big],$$

where the difference is not having

$$C\sup_{t\in[0,T]}\Big[P(\|v_t\|_{2.5}^2)+P(\|v\|_{3.5}^2+P(\|\eta\|_{4.5}^2)\Big]+CP(\|\sqrt{\kappa}v_{tt}\|^2_{L^2(0,T;H^{2.5}(\Omega))})$$

on the right-hand side of the inequality. To see this, for example, one such term comes from estimating

$$\sup_{t\in[0,T]}|P(v,\partial\eta_\kappa)|_{L^\infty}(\partial\Omega)\int_0^T|\sqrt{\kappa}\partial_t^3 v\cdot n_\kappa|_1|\sqrt{\kappa}\partial_t^2 v_\kappa|_2dt\,.$$

Since $P(v,\partial\eta_\kappa)_t\in L^\infty(0,T;L^1(\Gamma))$, by the fundamental theorem of calculus,

$$\sup_{t\in[0,T]}|P(v,\partial\eta_\kappa)|_{L^\infty(\Gamma)}\le M_0+CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))$$

and hence

$$\sup_{t\in[0,T]}|P(v,\partial\eta_\kappa)|_{L^\infty}\int_0^T|\sqrt{\kappa}\partial_t^3 v\cdot n_\kappa|_1|\sqrt{\kappa}\partial_t^2 v_\kappa|_2dt$$
$$\le\delta\sup_{t\in[0,T]}E_\kappa(t)+CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+C(\delta)\int_0^T\|\sqrt{\kappa}v_{tt}^+\|_{2.5}^2dt\,,$$

instead of having $\|\sqrt{\kappa}v_{tt}\|^4_{L^2(0,T;H^{2.5}(\Omega))}$ in the bound shown in [4]. Therefore, the energy estimates we cite from [4] will have only one polynomial type of term in the bound: $CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))$.

In this section, we will make use of the following equality which follows from (4.1e)

$$n_\kappa\cdot(v_{ttt}^+-v_{ttt}^-)=n_{\kappa ttt}\cdot(v^--v^+)-3n_{\kappa tt}\cdot(v_t^+-v_t^-)-3n_{\kappa t}\cdot(v_{tt}^+-v_{tt}^-)\,.\quad(8.3)$$

8.1. **Estimates for the third time-differentiated κ-problem.** Three time differentiate (8.2), and then use v_{ttt} as a test function and integrate in time from 0 to T, we find that

$$\sum_{\text{sign}=\pm}\int_0^T\int_{\Omega^{\text{sign}}}\rho^{\text{sign}}(\mathcal{J}_\kappa v_t^{\text{sign}i})_{ttt}v_{ttt}^{\text{sign}i}+\Big[(a_\kappa)_i^j q_{,j}^{\text{sign}}\Big]_{ttt}v_{ttt}^{\text{sign}i}dxdt$$
$$+\int_0^T\int_{\Omega^-}\rho^-\Big[(a_\kappa)_j^k(v^{-j}-v_e^{-j})v_{,k}^{-i}\Big]_{ttt}v_{ttt}^{-i}dxdt=0\,.$$

The terms needed additional analysis are

$$\mathcal{I}_1 = \int_0^T \int_{\Omega^-} \rho^- \Big[(a_\kappa)^k_j (v^{-j} - v_e^{-j}) v^{-i}_{,k}\Big]_{ttt} v^{-i}_{ttt} dxdt\,,$$
$$\mathcal{I}_2 = \int_0^T \int_{\Omega^+} \Big[(a_\kappa)^j_i q^+_{,j}\Big]_{ttt} v^{+i}_{ttt} dxdt + \int_0^T \int_{\Omega^-} \Big[(a_\kappa)^j_i q^-_{,j}\Big]_{ttt} v^{-i}_{ttt} dxdt\,.$$

The worst terms of $\mathcal{I}_1$ is when all the time derivatives hit $v_{,k}$, while the other combinations are bounded by $C\mathcal{P}(E_\kappa)$. Therefore,

$$\begin{aligned}\mathcal{I}_1 &\leq \int_0^T \int_{\Omega^-} \rho^- (a_\kappa)^k_j (v^{-j} - v_e^{-j}) v^{-i}_{ttt,k} v^{-i}_{ttt} dxdt + CT\mathcal{P}(E_\kappa)\\ &= \frac{1}{2}\int_0^T \int_{\Omega^-} \rho^- (a_\kappa)^k_j (v^{-j} - v_e^{-j}) |v_{ttt}|^2_{,k} dxdt + CT\mathcal{P}(E_\kappa)\\ &= -\frac{1}{2}\int_0^T \int_{\partial\Omega^-} \rho^- (a_\kappa)^k_j (v^{-j} - v_e^{-j}) N_k |v_{ttt}|^2 dSdt + CT\mathcal{P}(E_\kappa)\,.\end{aligned}$$

The boundary of Ω^- consists of Γ and $\partial\Omega$. On $\partial\Omega$, $a_\kappa = \mathrm{Id}$ and $v_e = 0$. Therefore, by (4.1f),

$$\int_{\partial\Omega} \rho^- (a_\kappa)^k_j (v^{-j} - v_e^{-j}) N_k |v_{ttt}|^2 dS = \int_{\partial\Omega} \rho^- (v^- \cdot N) |v_{ttt}|^2 dS = 0\,.$$

On Γ, since $v_e^- = v^+$ and $n_\kappa^j = g_\kappa^{-\frac{1}{2}} (a_\kappa)^k_j N_k$, boundary condition (4.1e) implies that

$$\int_{\Gamma} \rho^- (a_\kappa)^k_j (v^{-j} - v_e^{-j}) N_k |v_{ttt}|^2 dS = \int_\Gamma \sqrt{g_\kappa} \rho^- [v \cdot n_\kappa]_\pm |v_{ttt}|^2 dS = 0\,.$$

Therefore,

$$\mathcal{I}_1 \leq CT\mathcal{P}(E_\kappa)\,. \tag{8.4}$$

The worst terms of $\mathcal{I}_2$ is when all the time derivatives hit q. Therefore,

$$\begin{aligned}\mathcal{I}_2 &\leq \int_0^T \int_{\Omega^+} (a_\kappa)^j_i q^+_{ttt,j} v^{+i}_{ttt} dxdt + \int_0^T \int_{\Omega^-} (a_\kappa)^j_i q^-_{ttt,j} v^{-i}_{ttt} dxdt + CT\mathcal{P}(E_\kappa)\\ &= \sum_{\mathrm{sign}=\pm} \int_0^T \Big[\int_\Gamma q^{\mathrm{sign}}_{ttt} (a_\kappa)^j_i N_j v^{\mathrm{sign}i}_{ttt} dS - \int_{\Omega^{\mathrm{sign}}} q^{\mathrm{sign}}_{ttt} (a_\kappa)^j_i v^{\mathrm{sign}i}_{ttt,j} dx\Big] dt + CT\mathcal{P}(E_\kappa)\\ &= \mathcal{I}_{21} + \mathcal{I}_{22} + CT\mathcal{P}(E_\kappa)\,.\end{aligned}$$

For $\mathcal{I}_{21}$, it follows that

$$\begin{aligned}\mathcal{I}_{21} &= \int_0^T \int_\Gamma \sqrt{g_\kappa} n^i_\kappa \Big[q^+_{ttt} v^{+i}_{ttt} - q^-_{ttt} v^{-i}_{ttt}\Big] dSdt\\ &= \int_0^T \Big[\int_\Gamma \sqrt{g_\kappa} q^-_{ttt} (v^+_{ttt} - v^-_{ttt}) \cdot n_\kappa dS + \int_\Gamma \sqrt{g_\kappa} (q^+ - q^-)_{ttt} (v^+_{ttt} \cdot n_\kappa) dS\Big] dt\,.\end{aligned}$$

By (8.3) and substituting $-\sigma L_g(\eta_e)\cdot n_\kappa - \kappa\Delta_0(v\cdot n_\kappa)n_\kappa$ for $(q^+ - q^-)$, we apply the estimates as in [4] to obtain

$$\begin{aligned}
\mathcal{I}_{21} \le & -\int_0^T\int_\Gamma \sqrt{g_\kappa}q^-_{ttt}\Big[(v^+ - v^-)\cdot n_{\kappa ttt} + 3n_{\kappa t}\cdot(v^+_{tt} - v^-_{tt})\Big]dSdt \qquad (\equiv I_{21_a})\\
& - 3\int_0^T\int_\Gamma \sqrt{g_\kappa}q^-_{ttt}(v^+_t - v^-_t)\cdot n_{\kappa tt}dSdt \qquad (\equiv I_{21_b})\\
& + \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) + C(\delta)\Big[\|v^+_t\|^2_{2.5,+}\\
& + \|v^+\|^2_{3.5,+} + \|\eta_e\|^2_{4.5,+}\Big]\,.
\end{aligned}$$

Integrating by parts in time, since $\Big[\sqrt{g_\kappa}(v^+_t - v^-_t)\cdot n_{\kappa tt})\Big]_t \in L^\infty(0,T;L^2(\Gamma))$, using the same techniques as in [4], we find that

$$\begin{aligned}
\frac{\mathcal{I}_{21_b}}{3} = & \int_0^T\int_\Gamma q^-_{tt}\Big[\sqrt{g_\kappa}(v^+_t - v^-_t)\cdot n_{\kappa tt}\Big]_t dSdt - \int_\Gamma q^-_{tt}\sqrt{g_\kappa}(v^+_t - v^-_t)\cdot n_{\kappa tt}dS\Big|^{t=T}_{t=0}\\
\le & \ \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,. \qquad (8.5)
\end{aligned}$$

Let the first and the second term of $\mathcal{I}_{21_a}$ be denoted by $\mathcal{I}_{21_{a_1}}$ and $\mathcal{I}_{21_{a_2}}$, respectively. Integrating by parts in time,

$$\begin{aligned}
\frac{\mathcal{I}_{21_{a_2}}}{3} = & \int_0^T\int_\Gamma q^-_{tt}\Big[\sqrt{g_\kappa}n_{\kappa t}\cdot(v^+_{ttt} - v^-_{ttt}) + (\sqrt{g_\kappa}n_{\kappa t})_t\cdot(v^+_{tt} - v^-_{tt})\Big]dSdt\\
& - \int_\Gamma \sqrt{g_\kappa}q^-_{tt}n_{\kappa t}\cdot(v^+_{tt} - v^-_{tt})dS\Big|^{t=T}_{t=0}\,.
\end{aligned}$$

By (5.25), $\Big[\sqrt{g_\kappa}n_{\kappa t}\cdot(v^+_{tt} - v^-_{tt})\Big]_t \in L^2(0,T;H^{-0.5}(\Gamma))$. Since $q^-_{tt}\in L^\infty(0,T;H^{0.5}(\Gamma))$, it follows that

$$\int_\Gamma \sqrt{g_\kappa}q^-_{tt}n_{\kappa t}\cdot(v^+_{tt} - v^-_{tt})dS\Big|^{t=T}_{t=0} \le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,.$$

Again by (5.25), we can estimate the first integral of $\mathcal{I}_{21_{a_2}}$ and obtain

$$\mathcal{I}_{21_{a_2}} \le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,. \qquad (8.6)$$

For $\mathcal{I}_{21_{a_1}}$, integrating by parts in time again,

$$\begin{aligned}
\mathcal{I}_{21_{a_1}} = & \int_0^T\int_\Gamma q^-_{tt}\Big[\sqrt{g_\kappa}(v^+ - v^-)\cdot\partial_t^4 n_\kappa + [\sqrt{g_\kappa}(v^+ - v^-)]_t n_{\kappa ttt}\Big]dSdt\\
& - \int_\Gamma \sqrt{g_\kappa}q^-_{tt}(v^+ - v^-)\cdot n_{\kappa ttt}dS\Big|^{t=T}_{t=0}\,.
\end{aligned}$$

The second term of $\mathcal{I}_{21_{a_1}}$ can be bounded by $CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))$ since the integrand is in $L^\infty(0,T;L^1(\Gamma))$. Since $n_{\kappa ttt} \sim F_1(\partial\eta_\kappa)\partial v_{\kappa tt} + F_2(\partial\eta_\kappa,\partial v_\kappa)\partial v_{\kappa t}$, by the

fact that $\left[\sqrt{g_\kappa}(v^+ - v^-)(F_1 + F_2(\partial\eta_\kappa, \partial v_\kappa)\partial v_{\kappa t})\right]_t \in L^\infty(0,T;L^2(\Gamma))$ and $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality pairing,

$$
\begin{aligned}
&\int_\Gamma \sqrt{g_\kappa} q^-_{tt}(v^+ - v^-)\cdot n_{\kappa ttt} dS\Big|_{t=0}^{t=T} \\
&\leq M_0 + \Big[M_0 + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\Big]|q^-_{tt}(T)|_{0.5}\Big[|\partial v_{\kappa tt}(T)|_{-0.5} + 1\Big] \\
&\leq M_0 + M_0\|q^-_{tt}(T)\|_{1,-}\Big[\|v_{\kappa tt}(T)\|_{1,+} + 1\Big] + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) \\
&\leq M_0(\delta) + \delta \sup_{t\in[0,T]} E_\kappa(t) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,,
\end{aligned}
$$

where $\|v_{\kappa tt}(T)\|_{0,+} \leq M_0 + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))$ and Young's inequality are used to obtain the last inequality.

It remains to estimate the first term of $\mathcal{I}_{21_{a_1}}$ in order to complete the estimate of $\mathcal{I}_{21}$. We write the first term as

$$
\begin{aligned}
&-\int_0^T\int_\Gamma (q^+_{tt} - q^-_{tt})\sqrt{g_\kappa}(v^+ - v^-)\cdot \partial_t^4 n_\kappa dSdt \qquad (\equiv \mathcal{I}_3) \\
&+\int_0^T\int_\Gamma q^+_{tt}\sqrt{g_\kappa}(v^+ - v^-)\cdot \partial_t^4 n_\kappa dSdt\,. \qquad (\equiv \mathcal{I}_4)
\end{aligned}
$$

By $n_{\kappa t} = -g_\kappa^{\alpha\beta}(v_{\kappa,\alpha}\cdot n_\kappa)\eta_{\kappa,\beta}$,

$$
\begin{aligned}
\mathcal{I}_4 = &-\int_0^T\int_\Gamma q^+_{tt}\sqrt{g_\kappa}(v^+ - v^-)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta}(v_{\kappa ttt,\alpha}\cdot n_\kappa)dSdt \\
&+\int_0^T\int_\Gamma q^+_{tt} F(\partial\eta_\kappa, \partial v_\kappa, \partial v_{\kappa t})(\partial v_{\kappa tt}\cdot n_\kappa + 1)dSdt
\end{aligned}
$$

where the second integral is bounded by $CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))$. For the first term,

$$
\begin{aligned}
&\int_0^T\int_\Gamma q^+_{tt}\sqrt{g_\kappa}(v^+ - v^-)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta}(v_{\kappa ttt,\alpha}\cdot n_\kappa)dSdt \\
&=\int_0^T\int_\Gamma q^+_{tt}\sqrt{g_\kappa}(v^+ - v^-)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta}\Big[(v_{\kappa ttt}\cdot n_\kappa)_{,\alpha} - (v_{\kappa ttt}\cdot n_{\kappa,\alpha})\Big]dSdt\,.
\end{aligned}
$$

It follows from $H^{0.5}(\Gamma)$-$H^{-0.5}(\Gamma)$ duality pairing and (5.25) that the term with $(v_{\kappa ttt}\cdot n_{\kappa,\alpha})$ is also bounded by $CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))$.

Let ξ be a non-negative cut-off function so that $\mathrm{supp}\xi \subset \bigcup_i \mathrm{supp}\alpha_i$ and $\xi = 1$ on Γ. Integrating by parts in space, since $\partial\Gamma = \phi$, by the divergence theorem,

$$
\begin{aligned}
&\int_0^T\int_\Gamma q^+_{tt}\sqrt{g_\kappa}(v^+ - v^-)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta}(v_{\kappa ttt}\cdot n_\kappa)_{,\alpha}dSdt \\
&= -\int_0^T\int_\Gamma \Big[\xi\sqrt{g_\kappa}(v^+ - v^-)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta}\Big]_{,\alpha} q^+_{tt}(v_{\kappa ttt}\cdot n_\kappa)dSdt \quad (\equiv \mathcal{I}_{41}) \\
&\quad -\int_0^T\int_\Gamma q^+_{tt,\alpha}\xi(v^+ - v^-)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta} v^i_{\kappa ttt}(a_\kappa)^j_i N_j dSdt \\
&= \mathcal{I}_{41} - \int_0^T\int_{\Omega^+} (a_\kappa)^j_i\Big[\xi q^+_{tt,\alpha}(v^+ - w)\cdot \eta_{\kappa,\beta} g_\kappa^{\alpha\beta} v^i_{\kappa ttt}\Big]_{,j} dxdt\,, \qquad (8.7)
\end{aligned}
$$

where w is an $H^{5.5}$-extension of v^- to Ω^+. By (5.25), $\mathcal{I}_{41}$ can be bounded by $CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))$ as well. For the rest terms, there are two worst cases: when

the derivative ∂_j hits $q^+_{tt,\alpha}$ or $v^i_{\kappa ttt}$. For the latter case, by inequality (8.1),

$$\|\xi(a_\kappa)^j_i v^i_{\kappa ttt,j} - [\xi(a_\kappa)^j_i v^i_{ttt,j}]_\kappa\|_{0,+} \le C\kappa\|a_\kappa\|_{3,+}\|v_{ttt}\|_{1,+} \,.$$

This inequality together with the "divergence free" constraint implies

$$\|\xi(a_\kappa)^j_i v^i_{\kappa ttt,j}\|_{0.+} \le C\kappa\|a_\kappa\|_{3,+}\|v_{ttt}\|_{1,+} + C\mathcal{P}(E_\kappa(t))$$

and therefore by Young's inequality,

$$\begin{aligned}&\int_0^T\int_{\Omega^+}\xi(a_\kappa)^j_i q^+_{tt,\alpha}(v^+ - w)\cdot\eta_{\kappa,\beta}g^{\alpha\beta}_\kappa v^i_{\kappa ttt,j}dxdt\\ &\le \delta\sup_{t\in[0,T]}E_\kappa(t) + C(\delta)T\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\,.\end{aligned}$$

For the former case, we make use of the equation (4.1a) to substitute $(a_\kappa)^k_i q_{,k}$ for v^i_t. Therefore, in this case the worst term is

$$\int_0^T\int_{\Omega^+}\xi\partial_\alpha[(a_\kappa)^j_i q^+_{tt,j}](v^+ - w)\cdot\eta_{\kappa,\beta}g^{\alpha\beta}_\kappa[(a_\kappa)^k_i q^+_{tt,j}]_\kappa dxdt \equiv \int_0^T\int_{\Omega^+}\partial_\alpha Q_i Q_{i\kappa}F^\alpha dxdt\,,$$

Let $Q_i = (a_\kappa)^j_i q^+_{tt,j}$ and $F^\alpha = \xi(v^+ - w)\cdot\eta_{\kappa,\beta}g^{\alpha\beta}_\kappa$. By the definition of horizontal convolution by layers, we find that

$$\int_0^T\int_{\Omega^+}\partial_\alpha Q_i Q_{i\kappa}F^\alpha dxdt = \sum_\ell\int_0^T\int_{[0,1]^3}(\partial_\alpha Q_i)(\theta_\ell)[\rho\star_h\rho\star_h(Q_i(\theta_\ell))]F^\alpha(\theta_\ell)dydt\,.$$

Since $(\partial_\alpha Q_i)(\theta) = \Theta^\gamma_\alpha\partial_\gamma(Q_i(\theta))$,

$$\begin{aligned}&\int_0^T\int_{[0,1]^3}(\partial_\alpha Q_i)(\theta_\ell)[\rho\star_h\rho\star_h(Q_i(\theta_\ell))]F^\alpha(\theta_\ell)dxdt\\ &= \frac{1}{2}\int_0^T\int_{[0,1]^3}\partial_\gamma|\rho\star_h(Q_i(\theta_\ell))|^2F^\alpha(\theta_\ell)(\Theta_\ell)^\gamma_\alpha dydt + \int_0^T Rdt\,,\end{aligned}$$

where $R = \rho\star_h\Big[F^\alpha(\theta_\ell)(\Theta_\ell)^\gamma_\alpha\partial_\gamma Q_i(\theta_\ell)\Big] - F^\alpha(\theta_\ell)(\Theta_\ell)^\gamma_\alpha\rho\star_h\Big[\partial_\gamma Q_i(\theta_\ell)\Big]$ and by inequality (8.1), since $\nabla Q_i \sim F_1(\partial\eta_\kappa)v_{ttt} + F_2(\partial\eta_\kappa,\partial v_\kappa,\partial v_{\kappa t})\nabla q + F_3(\partial\eta_\kappa,\partial v_\kappa)\nabla q_t$,

$$\begin{aligned}\int_0^T|R|dt &\le C\kappa\int_0^T\|F(\theta)\Theta_\ell\|_{W^{1,\infty}([0,1]^3)}\|\partial(Q_i(\theta))\|_{L^2([0,1]^3)}dt\\ &\le M_0(\delta) + \delta\sup_{t\in[0,T]}E_\kappa(t) + CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\,.\end{aligned}$$

Integrating by parts in space,

$$\int_0^T\int_{[0,1]^3}\partial_\gamma|\rho\star_h(Q_i(\theta_\ell))|^2F^\alpha(\theta_\ell)(\Theta_\ell)^\gamma_\alpha dydt \le CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\,.$$

Combining all the estimates above, we find that

$$\mathcal{I}_4 \le M_0(\delta) + \delta\sup_{t\in[0,T]}E_\kappa(t) + C(\delta)T\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\,. \tag{8.8}$$

Now we turn our attention to $\mathcal{I}_{22}$ before estimating $\mathcal{I}_3$. By the "divergence free" constraint (4.1b),

$$\int_0^t\mathcal{I}_{22}dt = \sum_{\text{sign}=\pm}\int_0^t\int_{\Omega^{\text{sign}}}q^{\text{sign}}_{ttt}\Big[(a_{\kappa ttt})^j_i v^{\text{sign}i}_{,j} + 3(a_{\kappa tt})^j_i v^{\text{sign}i}_{t,j} + 3(a_{\kappa t})^j_i v^{\text{sign}i}_{tt,j}\Big]dxdt\,.$$

As shown in [4], it follows from integrating by parts in time that

$$\int_0^T \int_{\Omega^\pm} (a_{\kappa tt})^j_i v^{\pm i}_{t,j} q^\pm_{ttt} dx ds \le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) . \quad (8.9)$$

For the first and the third term, we follow [4] and obtain

$$\begin{aligned}
&\sum_{\text{sign}=\pm} \int_0^T \int_{\Omega^{\text{sign}}} q^{\text{sign}}_{ttt} \Big[(a_{\kappa ttt})^j_i v^{\text{sign}i}_{,j} + 3(a_{\kappa t})^j_i v^{\text{sign}i}_{tt,j} \Big] dx dt \\
\le &\sum_{\text{sign}=\pm} \int_0^T \int_{\Omega^{\text{sign}}} \mathcal{J}_\kappa^{-1} (a_\kappa)^r_s (a_\kappa)^j_i \Big[v^s_{\kappa tt,r} v^{\text{sign}i}_{,j} + 3 v^s_{\kappa,r} v^{\text{sign}i}_{tt,j} \Big] q^{\text{sign}}_{ttt} dx dt \qquad (\equiv \mathcal{I}_{22_a}) \\
&- \sum_{\text{sign}=\pm} \int_0^T \int_{\Omega^{\text{sign}}} \mathcal{J}_\kappa^{-1} (a_\kappa)^r_i (a_\kappa)^j_s \Big[v^s_{\kappa tt,r} v^{\text{sign}i}_{,j} + 3 v^s_{\kappa,r} v^{\text{sign}i}_{tt,j} \Big] q^{\text{sign}}_{ttt} dx dt \quad (\equiv \mathcal{I}_{22_b}) \\
&+ \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) + C(\delta) \Big[\|v^+_t\|^2_{2.5,+} + \|v^+\|^2_{3.5,+} \\
&+ \|\eta_e\|^2_{4.5,+} + \int_0^T \|\sqrt{\kappa} v^+_{tt}\|^2_{2.5,+} dt \Big] .
\end{aligned}$$

Using the "divergence free" constraint again,

$$\begin{aligned}
\mathcal{I}_{22_a} &= -3 \sum_{\text{sign}=\pm} \int_0^T \int_{\Omega^{\text{sign}}} \mathcal{J}_\kappa^{-1} (a_\kappa)^r_s v^s_{\kappa,r} \Big[(a_{\kappa tt})^j_i v^{\text{sign}i}_{,j} + 2(a_{\kappa t})^j_i v^{\text{sign}i}_{t,j} \Big] q^{\text{sign}}_{ttt} dx dt \\
&\le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) , \qquad (8.10)
\end{aligned}$$

where we apply estimates similar to (8.9) again from [4].

Integrating by parts in time (and space if there is $v_{\kappa ttt}$ or v_{ttt}), since $a_\kappa = \text{Id}$ on $\partial\Omega$ and $v_\kappa = 0$ outside Ω' (or near $\partial\Omega$), we find that

$$\begin{aligned}
\mathcal{I}_{22_b} \le &\int_0^T \int_\Gamma \mathcal{J}_\kappa^{-1} (a_\kappa)^r_s (a_\kappa)^j_i \Big[v^i_{\kappa ttt} v^{-s}_{,r} q^-_{tt} - v^i_{\kappa ttt} v^{+s}_{,r} q^+_{tt} \Big] N_j dS dt \qquad (\equiv \mathcal{I}_{22_{b_1}}) \\
&- 3 \int_0^T \int_\Gamma \mathcal{J}_\kappa^{-1} (a_\kappa)^r_s (a_\kappa)^j_i \Big[v^s_{\kappa,r} v^{+i}_{ttt} q^+_{tt} - v^s_{\kappa,r} v^{-i}_{ttt} q^-_{tt} \Big] N_j dS dt \quad (\equiv \mathcal{I}_{22_{b_2}}) \\
&+ \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) ,
\end{aligned}$$

where similar estimates for the lower order terms are obtained as those in [4]. It follows from (5.25) and (8.3) that

$$\mathcal{I}_{21_{b_1}} \le \int_0^T |v_{\kappa ttt}|_{-0.5} \mathcal{P}(E_\kappa(t)) dt \le CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) , \qquad (8.11)$$

$$\begin{aligned}
\mathcal{I}_{22_{b_2}} &= -3 \int_0^T \int_\Gamma \sqrt{g_\kappa} \mathcal{J}_\kappa^{-1} (a_\kappa)^r_s \Big[(v^+_{ttt} - v^-_{ttt}) \cdot n_\kappa q^-_{tt} + (q^+ - q^-)_{tt} (v^+_{ttt} \cdot n_\kappa) \Big] dS dt \\
&\le \delta \sup_{t\in[0,T]} E_\kappa(t) + M_0(\delta) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) , \qquad (8.12)
\end{aligned}$$

where we use the boundary condition (4.1c) in the second term and apply the same estimates as in [4].

For $\mathcal{I}_3$, we use the boundary condition (4.1d) in $\mathcal{I}_3$ and obtain

$$\begin{aligned}\mathcal{I}_3 &= -\int_0^T \int_\Gamma \Big[\sigma\Delta_g(\eta_e)\cdot n_\kappa + \kappa\Delta_0(v^+\cdot n_\kappa)\Big]_{tt}\sqrt{g_\kappa}(v^+-v^-)\cdot\partial_t^4 n_\kappa dSdt\\ &= \mathcal{I}_{31}+\mathcal{I}_{32}\,.\end{aligned}$$

The worst term of $\mathcal{I}_3$ is when the time derivatives hit the highest order term. Since $\displaystyle\int_0^T \Big[\|\sqrt{\kappa}v^+_{ttt}\|^2_{1.5,+} + \|\sqrt{\kappa}v^+_{tt}\|^2_{2.5,+}\Big]dt \le E_\kappa(T)$, by Young's inequality,

$$\mathcal{I}_{32} \le CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) + \int_0^T \Big[\delta\|\sqrt{\kappa}v^+_{ttt}\|^2_{1.5,+} + C(\delta)\|\sqrt{\kappa}v^+_{tt}\|^2_{2.5,+}\Big]dt. \tag{8.13}$$

Integrating by parts in time,

$$\begin{aligned}\mathcal{I}_{31} &= \int_0^T \int_\Gamma \sigma\Big[\Delta_g(\eta_e)\cdot n_\kappa\Big]_{ttt}\sqrt{g_\kappa}(v^+-v^-)\cdot\partial_t^3 n_\kappa dSdt \qquad (\equiv \mathcal{I}_{31_a})\\ &\quad - \int_\Gamma \sigma\Big[\Delta_g(\eta_e)\cdot n_\kappa\Big]_{tt}\sqrt{g_\kappa}(v^+-v^-)\cdot\partial_t^3 n_\kappa dS\Big|_{t=0}^{t=T} \qquad (\equiv \mathcal{I}_{31_b})\,.\end{aligned}$$

For $\mathcal{I}_{31_a}$, it follows from integration by parts (in space) that

$$\begin{aligned}\mathcal{I}_{31_a} \le &-\int_0^T \int_\Gamma \sigma g^{\gamma\delta}(v^+_{tt,\gamma})\cdot n_\kappa\sqrt{g_\kappa}(v^+-v^-)\cdot\eta_{\kappa,\beta}g_\kappa^{\alpha\beta}(v_{\kappa tt,\alpha\delta}\cdot n_\kappa)dSdt\\ &+ CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,.\end{aligned}$$

By the definition of v_κ, the inequality above implies that

$$\begin{aligned}\mathcal{I}_{31_a} \le &-\int_0^T \int_\Gamma \sigma[\partial_\gamma(\rho\star_h v^+_{tt})\cdot n_\kappa]F^{\alpha\gamma\delta}[\partial_\delta(\rho\star_h v^+_{tt})\cdot n_\kappa]_{,\alpha}dSdt\\ &+ CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,,\end{aligned}$$

where $F^{\alpha\gamma\delta} = \sqrt{g_\kappa}g_\kappa^{\alpha\beta}g^{\gamma\delta}(v^+-v^-)\cdot\eta_{\kappa,\beta}$. Since $F^{\alpha\gamma\delta}$ is symmetry in γ and δ, it follows from integration by parts that

$$\mathcal{I}_{31_a} \le \frac{1}{2}\int_0^T \int_\Gamma \sigma(\partial_\gamma v^+_{tt}\cdot n_\kappa)F^{\alpha\gamma\delta}_{,\alpha}(\partial_\delta v^+_{tt}\cdot n_\kappa)dSdt + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))$$

and hence

$$\mathcal{I}_{31_a} \le CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{8.14}$$

Integrating by parts in space, the worst term of $\mathcal{I}_{31_b}$ is

$$-\int_\Gamma \sigma F^{\alpha\gamma\delta}(\partial_\gamma v^+_t\cdot n_\kappa)(\partial_\delta v_{\kappa tt}\cdot n_\kappa)_{,\alpha}dS\,.$$

Since $F_t^{\alpha\gamma\delta} \in L^2(0,T;L^\infty(\Gamma))$, integrating by parts in space, we find that

$$\mathcal{I}_{31_b} \le \delta\sup_{t\in[0,T]} E_\kappa(t) + C(\delta)\|v^+_t\|^2_{2.5,+} + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t))\,. \tag{8.15}$$

Combining all the estimates above,

$$
\begin{aligned}
&\sup_{t\in[0,T]}\Big[\|v_{ttt}\|^2_{0,\pm}+|v^+_{tt}\cdot n|^2_1\Big]+\int_0^T|\sqrt{\kappa}v^+_{ttt}\cdot n_\kappa|^2_1dt\\
&\le M_0(\delta)+\delta\sup_{t\in[0,T]}E_\kappa(t)+C(\delta)T\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+C(\delta)\Big[\|v^+_t\|^2_{2.5,+} \qquad (8.16)\\
&\quad+\|v^+\|^2_{3.5,+}+\|\eta_e\|^2_{4.5,+}+\int_0^T\|\sqrt{\kappa}v^+_{tt}\|^2_{2.5,+}dt\Big]\,.
\end{aligned}
$$

We also need controls for $|v^-_{tt}\cdot n|_1$. It follows from inequality (8.1) and the fundamental theorem of calculus that

$$|w\cdot(n-n_\kappa)|_1\le C\kappa\Big[M_0+CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))\Big]|w|_1|\eta^+|_{4.5}\,.$$

Therefore, by (4.1e) and the fundamental theorem of calculus,

$$
\begin{aligned}
&|v^-_{tt}\cdot n|_1\le|v^-_{tt}\cdot n_\kappa|_1+|v^-_{tt}\cdot(n-n_\kappa)|_1\\
&\le|(v^+_{tt}-v^-_{tt})\cdot n_\kappa|_1+|v^+_{tt}\cdot n_\kappa|_1+|v^-_{tt}\cdot(n-n_\kappa)|_1\\
&\le|2(v^+_t-v^-_t)\cdot n_{\kappa t}+(v^+-v^-)\cdot n_{\kappa tt}|_1+|v^+_{tt}\cdot n|_1+|v_{tt}\cdot(n-n_\kappa)|_{1,\pm}\\
&\le M_0(\delta)+\delta\sup_{t\in[0,T]}E_\kappa(t)+CT\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+|v^+_{tt}\cdot n|_1\,.
\end{aligned}
$$

Having this additional inequality, we find that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\Big[\|v_{ttt}\|^2_{0,\pm}+|v_{tt}\cdot n|^2_{1,\pm}\Big]+\int_0^T|\sqrt{\kappa}v^+_{ttt}\cdot n_\kappa|^2_1dt\\
&\le M_0(\delta)+\delta\sup_{t\in[0,T]}E_\kappa(t)+C(\delta)T\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+C(\delta)\Big[\|v^+_t\|^2_{2.5,+} \qquad (8.17)\\
&\quad+\|v^+\|^2_{3.5,+}+\|\eta^+\|^2_{4.5,+}+\int_0^T\|\sqrt{\kappa}v^+_{tt}\|^2_{2.5,+}dt\Big]\,.
\end{aligned}
$$

8.2. **Estimates for the second time-differentiated κ-problem.** Similar to (12.33) in [4], let $\xi\partial\partial_t^2$ act on (4.1b) and test against $\xi\partial v_{tt}$, we find that for $\delta_1>0$,

$$
\begin{aligned}
&\sup_{t\in[0,T]}|\partial^2v_t\cdot n|^2_{0,\pm}+\int_0^T|\sqrt{\kappa}\partial^2v^+_{tt}\cdot n_\kappa|^2_{0,\pm}dt\le M_0(\delta_1) \qquad (8.18)\\
&\quad+\delta_1\sup_{t\in[0,T]}E_\kappa(t)+C(\delta_1)T\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+C(\delta_1)\int_0^T\|\sqrt{\kappa}v^+_t\|^2_{3.5,+}dt\,.
\end{aligned}
$$

8.3. **Estimates for the time-differentiated κ-problem.** Let $\xi\partial^2\partial_t$ act on (4.1b) and test against $\xi\partial^2v_t$, we find that for $\delta_2>0$,

$$
\begin{aligned}
&\sup_{t\in[0,T]}|\partial^3v\cdot n|^2_{0,\pm}+\int_0^T|\sqrt{\kappa}\partial^3v^+_t\cdot n_\kappa|^2_{0,\pm}dt\Big\}\le M_0(\delta_2) \qquad (8.19)\\
&\quad+\delta_2\sup_{t\in[0,T]}E_\kappa(t)+C(\delta_2)T\mathcal{P}(\sup_{t\in[0,T]}E_\kappa(t))+C(\delta_2)\int_0^T\|\sqrt{\kappa}v^+\|^2_{4.5,+}dt\,.
\end{aligned}
$$

8.4. **The third tangential space differentiated κ-problem.** Similar to (12.37) in [4], the study of the boundary condition (4.1d) leads to the following important elliptic estimate:

$$\sup_{t\in[0,T]} |\sqrt{\kappa}\eta^+(t)|^2_{5,\pm} \le M_0 + C \sup_{t\in[0,T]} E_\kappa(t) + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) . \tag{8.20}$$

Let $\xi\partial^3$ act on (4.1) and test against $\xi\partial^3 v$, by (8.20), we find that for $\delta_3 > 0$,

$$\begin{aligned} &\sup_{t\in[0,T]} |\partial^4\eta^+ \cdot n|^2_{0,\pm} + \int_0^T |\sqrt{\kappa}\partial^4\eta^+ \cdot n_\kappa|^2_{0,\pm} dt \\ &\le M_0(\delta_3) + \delta_3 \sup_{t\in[0,T]} E_\kappa(t) + C(\delta_3)T\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) . \end{aligned} \tag{8.21}$$

8.5. **A polynomial-type inequality for the energy and the existence of solutions.** Combining the div-curl estimates (5.23), (5.24), the energy estimates (8.17), (8.18), (8.19), (8.21), we find that

$$\begin{aligned} E_\kappa(t) \le M_0(\delta,\delta_1,\delta_2,\delta_3) + (\delta + \delta_1 C(\delta) + \delta_2 C(\delta_1) + \delta_3 C(\delta_2)) \sup_{t\in[0,T]} E_\kappa(t) \\ + C(\delta,\delta_1,\delta_2,\delta_3)T\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) . \end{aligned}$$

Choose $\delta > 0$ and $\delta_j > 0$ small enough so that $\delta + \delta_1 C(\delta) + \delta_2 C(\delta_1) + \delta_3 C(\delta_2) \le \dfrac{1}{2}$, then the inequality above implies

$$\sup_{t\in[0,T]} E_\kappa(t) \le M_0 + CT\mathcal{P}(\sup_{t\in[0,T]} E_\kappa(t)) . \tag{8.22}$$

Therefore, there exists $T_1 > 0$ independent of κ so that

$$\sup_{t\in[0,T_1]} E_\kappa(t) \le 2M_0 . \tag{8.23}$$

This κ-independent estimate guarantees the existence of a solution to problem (1.1) by passing $\kappa \to 0$.

8.6. **Removing the additional regularity assumptions on the initial data.** In the previous sections, we in fact assume that v is smooth enough so that we can directly differentiate the Euler equation (4.1b) and test with suitable test functions. This requires higher regularity of the initial data, namely, $u_0^\pm \in H^{10.5}(\Omega^\pm)$ and $\Gamma \in H^7$. As in [4], this can be achieved by mollifying the interface by the horizontal convolution by layers and mollifying the initial velocity by the usual Fredrich's mollifiers.

8.7. **A posteriori elliptic estimates.** As in [4], by exactly the same proof, we find that for T sufficiently small,

$$\sup_{t\in[0,T]} [|\Gamma(t)|_{5.5} + \|v\|_{4.5,\pm} + \|v_t\|_{3,\pm}] \le \mathcal{M}_0 , \tag{8.24}$$

where $\mathcal{M}_0$ is some polynomial of M_0.

9. Optimal regularity for the initial data

In the previous discussion, the existence of the solution requires the initial data $u_0^\pm \in H^{4.5}(\Omega^\pm)$. We show that this requirement can be loosened to $u_0^\pm \in H^3(\Omega^\pm)$ and $\Gamma \in H^4$ in this section, by assuming that we already have a solution to the problem.

In this section, we study the problem in the Eulerian framework. To start the argument, we define the energy function $\mathcal{E}(t)$ first. Let $\mathcal{E}(t)$ be define by

$$\begin{aligned}\mathcal{E}(t) = |\Gamma(t)|_4^2 + \|u^+\|^2_{H^3(\Omega^+(t))} + \|u^-\|^2_{H^3(\Omega^-(t))} + \|u_t^+\|^2_{H^{1.5}(\Omega^+(t))}\\ + \|u_t^-\|^2_{H^{1.5}(\Omega^-(t))} + \|u_{tt}^+\|^2_{L^2(\Omega^+(t))} + \|u_{tt}^-\|^2_{L^2(\Omega^-(t))}\,.\end{aligned}$$

Then for the pressure function $p^\pm$, we have the following estimate:

$$\|p^+\|^2_{H^{2.5}(\Omega^+(t))} + \|p^-\|^2_{H^{2.5}(\Omega^-(t))} + \|p_t^+\|^2_{H^1(\Omega^+(t))} + \|p_t^-\|^2_{H^1(\Omega^-(t))} \le C\mathcal{P}(\mathcal{E}(t))\,.$$

The estimates for $\operatorname{curl} u^\pm$ are essentially identical, while the estimates for $\operatorname{div} u^\pm$ are trivial because of the divergence free constraint (1.1b). Therefore,

$$\begin{aligned}\sup_{t\in[0,T]}\Big[&\|\operatorname{curl} u^+\|^2_{H^{2.5}(\Omega^+(t))} + \|\operatorname{curl} u^-\|^2_{H^{2.5}(\Omega^-(t))} + \|\operatorname{curl} u_t^+\|^2_{H^1(\Omega^+(t))}\\ &+ \|\operatorname{curl} u_t^-\|^2_{H^1(\Omega^-(t))}\|\operatorname{div} u^+\|^2_{H^{2.5}(\Omega^+(t))} + \|\operatorname{div} u^-\|^2_{H^{2.5}(\Omega^-(t))}\\ &+ \|\operatorname{div} u_t^+\|^2_{H^1(\Omega^+(t))} + \|\operatorname{div} u_t^-\|^2_{H^1(\Omega^-(t))}\Big]\\ \le M_0(\delta) + \delta &\sup_{t\in[0,T]}\mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]}\mathcal{E}(t))\end{aligned} \tag{9.1}$$

where $M_0(\delta) = M_0(|\Gamma|_4^2, \|u_0^+\|^2_{3,+}, \|u_0^-\|^2_{3,-}, \delta)$.

Remark 7. *The reason for not analyzing the problem in the ALE formulation is that in the minus region, the transported velocity $a^T(v^- - v_e^-)$ is only as regular as $\nabla\eta^+$, which is less regular than the velocity $v^\pm$. This prevents from obtaining the estimates for $\operatorname{curl} v^\pm$ in $H^{2.5}(\Omega^\pm)$. With the Eulerian formulation, the transport velocity $u^\pm$ is $H^3(\Omega^\pm(t))$, and the analysis goes through.*

Note that the need of that η is more regular than u (or v) is only for the study of the κ-problem, where the estimate of the boundary integrals with artificial viscosity κ requires that η_κ is as regular as $\sqrt{\kappa}v$ (which is "more regular" than v by the definition of the energy function E_κ). This observation implies that without worrying about the κ-terms, the energy estimates still follow. Therefore, by the identities

$$\frac{d}{dt}\int_{\Omega^\pm(t)} f(y,t)dy = \int_{\Omega^\pm(t)} (f_t + \nabla_{u^\pm} f)(y,t)dy$$

and

$$\int_{\Gamma(t)} f(y,t)dS_y = \int_\Gamma f(\eta(x,t),t)\sqrt{g}dS_x\,,$$

we can show, as shown in the previous sections, that

$$\begin{aligned}\sup_{t\in[0,T]}&\Big[\|u_{tt}^+\|^2_{L^2(\Omega^+(t))} + \|u_{tt}^-\|^2_{L^2(\Omega^-(t))} + |\partial^2 v\cdot n|^2_{0,\pm} + |\partial v_t\cdot n|^2_{0,\pm}\Big]\\ \le M_0(\delta) + \delta &\sup_{t\in[0,T]}\mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]}\mathcal{E}(t))\,,\end{aligned} \tag{9.2}$$

where the interior estimates are for the Eulerian velocity $u^\pm$ while the boundary estimates are for the ALE velocity $v^\pm$.

In addition to $|\Gamma(t)|_4^2$, it suffices to establish bounds for $|\partial^2 u^\pm \cdot m|^2_{H^{0.5}(\Gamma(t))}$ and $|\partial u^\pm_t \cdot m|^2_{L^2(\Gamma(t))}$, where m denotes the unit outward normal of $\Omega^+(t)$. We remark here that we use different notations to distinguish the "normal" on Γ and the normal on $\Gamma(t)$. In general, $n = m \circ \eta$.

The bounds for $|\partial u^\pm_t \cdot m|^2_{L^2(\Gamma(t))}$ follows from the energy estimate (9.2). Since

$$u^{\pm i}_{t,j} m^i = \Big[a^k_j v^{\pm i}_{t,k} n^i - a^k_j (v^{\pm m} a^\ell_m v^{\pm i}_{,\ell})_{,k} n^i\Big] \circ \eta^{-1} \quad \text{on} \ \ \Gamma(t)\,,$$

multiplying $\tau^j_\alpha = \Big(\frac{\eta^j_{,\alpha}}{|\eta_{,\alpha}|}\Big) \circ \eta^{-1}$ on both side, by $\|\delta - a\|_{2,+} \sim \mathcal{O}(t)$ we find that

$$\begin{aligned} |\partial_\alpha u^\pm_t \cdot m|_{L^2(\Gamma(t))} &\le C\Big[|\partial_\alpha v^\pm_t \cdot n|_0 + |\partial_\alpha (v^{\pm\ell} v^{\pm i}_{,\ell}) n^i|_0\Big] + CT\mathcal{P}(\sup_{t\in[0,T]} \mathcal{E}(t)) \\ &\le M_0(\delta) + \delta \sup_{t\in[0,T]} \mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]} \mathcal{E}(t))\,. \end{aligned} \tag{9.3}$$

For the bound of $|\partial^2 u^\pm \cdot m|^2_{H^{0.5}(\Gamma(t))}$, we first estimate $|\partial^2 v^\pm \cdot n|^2_{0.5}$. Similar to the a posteriori estimate in [4], by studying the boundary condition

$$\partial_t[(p^+ - p^-)\circ\eta^+ \cdot n] = -[\sqrt{g} g^{\alpha\beta} \Pi^i_j v^{+j}_{,\beta} + \sqrt{g}(g^{\nu\mu} g^{\alpha\beta} - g^{\alpha\nu} g^{\mu\beta}) \eta^i_{,\beta} \eta^j_{,\nu} v^{+j}_{,\mu}]_{,\alpha}\,,$$

where η and g are formed from v^+, we find that

$$|\partial^2 v^+ \cdot n|_0^2 \le M_0(\delta) + \delta |v^+|_2^2 + C\Big[\mathcal{P}(|\Gamma|^2_{3.5}, \mathcal{E}(t)) |\eta^+ - \mathrm{Id}|_2^2 + |p^+_t - p^-_t|_0^2\Big]\,,$$

$$|\partial^2 v^+ \cdot n|_1^2 \le M_0(\delta) + \delta |v^+|_3^2 + C\Big[\mathcal{P}(|\Gamma|^2_{3.5}, \mathcal{E}(t)) |\eta^+ - \mathrm{Id}|_3^2 + |p^+_t - p^-_t|_1^2\Big]\,,$$

and hence by interpolations,

$$\begin{aligned} \sup_{t\in[0,T]} |\partial^2 v^+ \cdot n|^2_{0.5} &\le M_0(\delta) + \delta \sup_{t\in[0,T]} \mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]} \mathcal{E}(t)) \\ &\quad + C\Big[\|p^+_t\|^2_{H^1(\Omega^+(t))} + \|p^-_t\|^2_{H^1(\Omega^-(t))}\Big]\,. \end{aligned}$$

By the elliptic problem

$$\begin{aligned} \Delta p^\pm_t &= 2\nabla u^\pm_t : (\nabla u^\pm)^T && \text{in} \ \ \Omega^\pm(t)\,, \\ \frac{\partial p^\pm_t}{\partial n} &= (u^\pm_{tt} + \nabla_{u^\pm_t} u^\pm + \nabla_{u^\pm} u^\pm_t)\cdot n && \text{on} \ \ \Gamma(t)\,, \end{aligned}$$

we find that

$$\begin{aligned} \|p^\pm_t\|^2_{H^1(\Omega^\pm(t))} &\le C\Big[\|\nabla u^\pm_t : (\nabla u^\pm)^T\|^2_{H^{0.5}(\Omega^\pm(t))} + \|u^\pm_{tt} + \nabla_{u_t} u + \nabla_u u_t\|^2_{L^2(\Omega^\pm(t))}\Big] \\ &\le M_0(\delta) + \delta \sup_{t\in[0,T]} \mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]} \mathcal{E}(t))\,, \end{aligned} \tag{9.4}$$

where we use (9.2) to estimate $\|u^\pm_{tt}\|^2_{L^2(\Omega^\pm(t))}$ and Young's inequality for the other terms. Therefore,

$$\sup_{t\in[0,T]} |\partial^2 v^+ \cdot n|^2_{0.5} \le M_0(\delta) + \delta \sup_{t\in[0,T]} \mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]} \mathcal{E}(t))\,.$$

By similar argument of obtaining (9.3), we find that

$$\sup_{t\in[0,T]} |\partial^2 u^+ \cdot m|^2_{H^{0.5}(\Omega^+(t))} \le M_0(\delta) + \delta \sup_{t\in[0,T]} \mathcal{E}(t) + CT\mathcal{P}(\sup_{t\in[0,T]} \mathcal{E}(t))\,. \tag{9.5}$$

The estimate of $|\partial^2 u^- \cdot m|^2_{H^{0.5}(\Gamma(t))}$ follows from the boundary condition (1.1d), as discussed in the previous sections.

It then follows from (9.1), (9.2), (9.3) and (9.5) that

$$\sup_{t\in[0,T]}\Big[\mathcal{E}(t)-|\Gamma(t)|_4^2\Big]\le M_0(\delta)+\delta\sup_{t\in[0,T]}\mathcal{E}(t)+CT\mathcal{P}(\sup_{t\in[0,T]}\mathcal{E}(t))\,. \tag{9.6}$$

With this estimate in mind, we can estimate $\|p^\pm\|^2_{H^{2.5}(\Omega^\pm(t))}$ in the way we obtain (9.4) and find that $\|p^\pm\|^2_{H^{2.5}(\Omega^\pm(t))}$ satisfies the same inequality. Let h be the height function of $\Gamma(t)$ over Γ. By exactly the same argument as in [4],

$$\sup_{t\in[0,T]}|h(t)|^2_{H^4}\le M_0(\delta)+\delta\sup_{t\in[0,T]}\mathcal{E}(t)+CT\mathcal{P}(\sup_{t\in[0,T]}\mathcal{E}(t))\,,$$

and hence

$$\sup_{t\in[0,T]}|\Gamma(t)|_4^2\le M_0(\delta)+\delta\sup_{t\in[0,T]}\mathcal{E}(t)+CT\mathcal{P}(\sup_{t\in[0,T]}\mathcal{E}(t))\,. \tag{9.7}$$

Combining (9.6) and (9.7), by choosing $\delta>0$ small enough, we obtain the same polynomial-type inequality as (8.22), and therefore there exists a $T>0$ so that

$$\sup_{t\in[0,T]}\mathcal{E}(t)\le 2M_0\,.$$

This proves the claim of the optimal regularity of the initial data to obtain the solution to (1.1).

Remark 8. *The argument in this section can also be used to prove the existence theorem for the one phase problem studied in* [4], *provided the same regularity of the initial velocity u_0 and the initial interface Γ are given.*

10. Uniqueness of solutions

Suppose that (v^1,q^1) and (v^2,q^2) are both solutions to (4.1) (with $\kappa=0$, $\mathcal{J}=1$) with initial data $u_0^\pm\in H^6(\Omega^\pm)$ and $\Gamma\in H^7$. Let η_e^1 and η_e^2 be defined as in Section 4 (with associated cofactor matrices a^1 and a^2), and set

$$\mathcal{E}_j(t)=\|\Gamma(t)\|_7^2+\sum_{k=0}^4\|\partial_t^k v^j(t)\|^2_{6-1.5k,\pm}+\sum_{k=0}^3\|\partial_t^k q^j(t)\|^2_{5.5-1.5k,\pm}\,.$$

By the existence theorem, both $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ are bounded by a constant $\mathcal{M}_0$ depending on the data u_0 and Γ on a time interval $0\le t\le T$ for T small enough.

Let $w=v^1-v^2$, $w_e=M^+w^+$ with associate flow map $\zeta_e=\int_0^t M^+w^+ds$, and $r=q^1-q^2$. The goal in this section is to show that $w=0$ by showing that the energy function

$$E(t)=\|v(t)\|^2_{3,\pm}+\|v_t(t)\|^2_{1.5,\pm}+\|v_{tt}(t)\|^2_{0,\pm}$$

is actually zero for a short time.

10.1. **The divergence and curl estimates.**

In Ω^+, v^{1+} and v^{2+} satisfy

$$\rho^+v_t^{+j}+a_j^\ell q_{,\ell}=0\qquad\text{for }\ (v,q)=(v^1,q^1)\text{ or }(v,q)=(v^2,q^2)\,.$$

Let $\varepsilon_{ijk}a_j^r\nabla_r$ act on both sides of the equality above and form the difference of the two equations, after integrating in time from 0 to t, we find that

$$\begin{aligned}\rho^+\operatorname{curl}w^{+i}(t)&=\varepsilon_{ijk}\int_0^t\Big[(a^1-a^2)_k^\ell(v_t^1)^{+j}_{,\ell}+[(a^2)_k^\ell-\delta_k^\ell]w^{+j}_{t,\ell}\Big]ds\\&=\varepsilon_{ijk}[(a^2)_k^\ell(t)-\delta_k^\ell]w^{+j}_{,\ell}(t)+\varepsilon_{ijk}\int_0^t\Big[(a^1-a^2)_k^\ell(v_t^1)^{+j}_{,\ell}-(\partial_ta^2)_k^\ell w^{+j}_{,\ell}\Big]ds\,.\end{aligned}$$

Therefore, by $\|a^2(t) - \delta(t)\|_{3.5,+} \le CT$,

$$\sup_{t\in[0,T]} \|\operatorname{curl} w^+(t)\|^2_{2,+} \le CT \sup_{t\in[0,T]} \|w^+(t)\|^2_{3,+}\,,$$

where C depends on $\mathcal{M}_0$ only. By the "divergence free" constraint $a^j_i v^i_{,j} = 0$, we similarly have

$$\sup_{t\in[0,T]} \|\operatorname{div} w^+(t)\|^2_{2,+} \le CT \sup_{t\in[0,T]} \|w^+(t)\|^2_{3,+}\,.$$

For the divergence and curl estimates in Ω^-, let $\tilde v^1$ and $\tilde v^2$ denote the Lagrangian velocity in Ω^-, that is,

$$\tilde v^j = \partial_t \tilde\eta^j = u^j \circ \tilde\eta^j \qquad \text{in } \ \Omega^-\,,$$

where u^j is the Eulerian velocity in Ω^-. The same argument as above shows that

$$\sup_{t\in[0,T]} \Big[\|\operatorname{curl} \tilde w(t)\|^2_{2,-} + \|\operatorname{div} \tilde w(t)\|^2_{2,-}\Big] \le CT \sup_{t\in[0,T]} \|\tilde w(t)\|^2_{3,-}\,, \tag{10.1}$$

where $\tilde w = \tilde v^1 - \tilde v^2$. We now convert (10.1) to the inequality with w replacing $\tilde w$.

Let $\zeta^j = (\tilde\eta^j)^{-1} \circ (\eta^j_e)^{-1}$ and $b^j = \nabla \zeta^j$ for $j = 1, 2$. Then

$$\begin{aligned}
\|\operatorname{curl} w(t)\|^2_{2,+} &= \sum_{i=1}^{\mathfrak{n}} \|\varepsilon_{ijk}[\tilde v^1 \circ \zeta^1 - \tilde v^2 \circ \zeta^2]^j_{,k}\|^2_{2,-} = \sum_{i=1}^{\mathfrak{n}} \|\varepsilon_{ijk}[(b^1)^r_k \tilde v^{1j}_{,r} - (b^2)^r_k \tilde v^{2j}_{,r}]\|^2_{2,-} \\
&= \sum_{i=1}^{\mathfrak{n}} \|\varepsilon_{ijk}[(b^1 - b^2)^r_k \tilde v^{1j}_{,r} + (b^2 - \delta)^r_k (\tilde v^1 - \tilde v^2)^j_{,r} + \tilde w^j_{k}]\|^2_{2,-} \\
&\le CT \sup_{t\in[0,T]} \Big[\|w^+\|^2_{3,+} + \|\tilde w(t)\|^2_{3,-}\Big]\,,
\end{aligned}$$

where we use $\|(b^1 - b^2)(t)\|^2_{2,-} \le CT \sup_{t\in[0,T]} \Big[\|w^+\|^2_{3,+} + \|\tilde w\|^2_{3,-}\Big]$ by studying the time derivative of $b^1 - b^2$. Similar argument shows that

$$\begin{aligned}
\|\operatorname{div} w\|^2_{3,-} &\le CT \sup_{t\in[0,T]} \Big[\|w^+\|^2_{3,+} + \|\tilde w\|^2_{3,-}\Big]\,, \\
\|\tilde w\|^2_{3,-} &\le CT \sup_{t\in[0,T]} \Big[\|w^+\|^2_{3,+} + \|\tilde w\|^2_{3,-}\Big]\,.
\end{aligned}$$

Thus for $T > 0$ small enough, we find that

$$\sup_{t\in[0,T]} \Big[\|\operatorname{curl} w(t)\|^2_{2,-} + \|\operatorname{div} w(t)\|^2_{2,-}\Big] \le CT \sup_{t\in[0,T]} \|w(t)\|^2_{3,-}\,.$$

The estimates for the divergence and curl of w_t are similar, so we omit here. In a nut shell,

$$\begin{aligned}
\sup_{t\in[0,T]} \Big[&\|\operatorname{curl} w(t)\|^2_{2,\pm} + \|\operatorname{div} w(t)\|^2_{2,\pm} + \|\operatorname{curl} w_t(t)\|^2_{0.5,\pm} \\
&+ \|\operatorname{div} w_t(t)\|^2_{0.5,\pm}\Big] \le CT \sup_{t\in[0,T]} E(t)\,.
\end{aligned} \tag{10.2}$$

REMARK 9. *We cannot obtain estimate (10.2) by studying the equations for v^- (with the transport velocity) directly, since it also requires the study of ζ^+_e as we did in the estimates of the κ-problem. This requires $u^\pm_0 \in H^5(\Omega^\pm)$ at least.*

10.2. **The boundary estimates.** First we note that (w, r) satisfies

$$
\begin{aligned}
\rho^{\pm} w_t^{\pm i} + (a^1)^\ell_j (v^{1-} - v_e^{1-})^j w^{-i}_{,\ell} + (a^1)^j_i r^{\pm}_{,j} &= F^{\pm} && \text{in } [0,T]\times\Omega^{\pm}\,, && (10.3a)\\
(a^1)^j_i w^{\pm i}_{,j} &= (a^2 - a^1)^j_i v^{2\pm i}_{,j} && \text{in } [0,T]\times\Omega^{\pm}\,, && (10.3b)\\
(r^+ - r^-) n_1 &= -\sigma \Pi^1 g^{1\alpha\beta} \zeta_{e,\alpha\beta} + B && \text{on } [0,T]\times\Gamma\,, && (10.3c)\\
w^+ \cdot n_1 &= w^- \cdot n_1 + b && \text{on } [0,T]\times\Gamma\,, && (10.3d)\\
w^- \cdot n_1 &= 0 && \text{on } [0,T]\times\partial\Omega\,, && (10.3e)\\
w^{\pm}(0) &= 0 && \text{in } \{t=0\}\times\Omega^{\pm}\,, && (10.3f)
\end{aligned}
$$

where

$$
\begin{aligned}
F^{\pm} &= \Big[(a^2 - a^1)^\ell_j (v^{2-} - v_e^{2-})^j - (a^1)^\ell_j (w^{-j} - w_e^{-j})\Big](v^{2-}_{,\ell})^i + (a^2 - a^1)^\ell_j q^{2\pm}_{,\ell}\,,\\
B &= -\sigma \Delta_{g^1 - g^2}(\eta_e^2) + (q^{2+} - q^{2-})(n_2 - n_1)\,,\\
b &= (v^{2+} - v^{2-})\cdot(n_2 - n_1)\,.
\end{aligned}
$$

The main difference between (10.3) and the uniqueness argument for the one phase problem (see Section 15 in [4]) is on the additional term b. In order to obtain estimate similar to (8.22) (except that in the uniqueness proof, we only study the second time differentiated problem), we need to estimate the integral

$$
\int_0^T \int_\Gamma r^-_{tt}(w^+_{tt} - w^-_{tt})\cdot n_1 dSdt\,.
$$

By (10.3e),

$$
(w^+_{tt} - w^-_{tt})\cdot n = b_{tt} - 2(w^+_t - w^-_t)\cdot n_t - (w^+ - w^-) n_{tt}\,.
$$

The only term we need to worry about is the integral with integrand $r^-_{tt}(v^{2+} - v^{2-})\cdot(n_2 - n_1)_{tt}$. The worst term of this integral is (after integration by parts in time)

$$
\begin{aligned}
&-\int_0^T \int_\Gamma r^-_{tt}\Big[g^{1\alpha\beta}(v^{1+}_{t,\alpha}\cdot n_1)\eta^1_{e,\beta} - g^{2\alpha\beta}(v^{2+}_{t,\alpha}\cdot n_2)\eta^2_{e,\beta}\Big] dSdt\\
&= -\int_\Gamma r^-_t\Big[g^{1\alpha\beta}(v^{1+}_{t,\alpha}\cdot n_1)\eta^1_{e,\beta} - g^{2\alpha\beta}(v^{2+}_{t,\alpha}\cdot n_2)\eta^2_{e,\beta}\Big] dS\Big|_{t=0}^{t=T}\\
&\quad + \int_0^T \int_\Gamma r^-_t\Big[g^{1\alpha\beta}(v^{1+}_{t,\alpha}\cdot n_1)\eta^1_{e,\beta} - g^{2\alpha\beta}(v^{2+}_{t,\alpha}\cdot n_2)\eta^2_{e,\beta}\Big]_t dSdt\,.
\end{aligned}
$$

Add and subtract terms to form the integrand in terms of w, $\eta^1_e - \eta^2_e$, $n_1 - n_2$ or $g^1 - g^2$. By Young's inequality, the first term (time boundary term) is bounded by $(\delta + C(\delta)T)\sup_{t\in[0,T]} E(t)$, where $C(\delta)$ depends on δ and $\mathcal{M}_0$. For the second term (time interior term), the worse term occurs when time differentiating ∂v_t. For this worst case, we can transform the surface integral to the interior interior using the divergence theorem as we did in (8.7). Therefore,

$$
\int_0^T \int_\Gamma r^-_t\Big[g^{1\alpha\beta}(v^{1+}_{t,\alpha}\cdot n_1)\eta^1_{e,\beta} - g^{2\alpha\beta}(v^{2+}_{t,\alpha}\cdot n_2)\eta^2_{e,\beta}\Big]_t dSdt \le (\delta + C(\delta)T)\sup_{t\in[0,T]} E(t)\,.
$$

The estimates with the addition of the forcing F, the right-hand side of (10.3c), and B is already done in [4]. It suffices to show that $|\partial^2 w\cdot n_1|_{0.5}$ has the same bound. However, since

$$
|B_t|^2_{0.5} \le CT \sup_{t\in[0,T]} E(t) + C|w|^2_{1.5} \le (\delta + C(\delta)T)\sup_{t\in[0,T]} E(t)\,,
$$

by study the first time derivative of (10.3c), similar to the a posteriori estimate, we find that

$$\sup_{t\in[0,T]} |\partial^2 w \cdot n_1|^2_{0.5} \le (\delta + C(\delta)T) \sup_{t\in[0,T]} E(t)\,.$$

Therefore, with (10.2) we conclude that $E(t)$ satisfies

$$\sup_{t\in[0,T]} E(t) \le (\delta + C(\delta)T) \sup_{t\in[0,T]} E(t)\,,$$

which implies for T small enough, $E(t) = 0$ and hence $w = 0$. In other words, we establish the uniqueness of the solution to the problem.

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